**COMPLEX ANALYSIS**
**WORKSHEEET 2**
**Instructor: G. Smyrlis**

1. Compute the following complex integrals:

(i) $\displaystyle\int_\gamma (3z^2 - 2z)dz, \quad \gamma(t) = t + it^2, \;\; t \in [0,1].$

(ii) $\displaystyle\int_\Gamma \mathrm{Im}(z - i)dz, \quad \Gamma = \gamma + [i, -1], \;\; \gamma(t) = e^{it}, \;\; t \in [0, \pi/2].$

(iii) $\displaystyle\int_\gamma \cos z dz, \quad \gamma = \Big[-\frac{\pi}{2} + i, \; \pi + i\Big].$

(iv) $\displaystyle\int_\gamma \frac{\mathrm{Log} z}{z} dz, \quad \gamma = [1, \; i].$

(vi) $\displaystyle\int_\gamma |z+1|^2 dz, \quad \gamma(t) = e^{it}, \;\; t \in [0, 2\pi].$

(vii) $\displaystyle\int_\gamma \overline{z^2 e^z} dz, \quad \gamma(t) = e^{it}, \;\; t \in [0, \pi].$

2. Prove:

(i) $\displaystyle\left|\int_\gamma \frac{e^{iz}}{z^2} \, dz\right| \; \leq \; \pi, \quad \text{where} \;\; \gamma(t) = e^{it}, \;\; t \in [0, \pi].$

(iii) $\displaystyle\left|\int_\gamma \frac{dz}{\overline{z}^2 + \overline{z} + 1}\right| \; \leq \; \frac{3\pi}{10}, \quad \text{where} \;\; \gamma(t) = 3e^{it}, \;\; t \in [0, \; \pi/2].$

3. Prove:

(i) $|\sin(z^2)| \; \leq \; e, \quad$ for all $z \in \mathbb{C}$ with $|z| = 1$.

(ii) $\displaystyle\left|\int_\gamma e^{2\overline{z}} \sin(z^2) \, dz\right| \; \leq \; 2\pi e^3, \;\; \text{where} \;\; \gamma(t) = e^{it}, \;\; t \in [0, \; 2\pi].$

4. Show that
$$\lim_{R\to+\infty}\int_{\gamma_R}\frac{e^{iz^2}}{1+z^2}dz=0,$$
where $\gamma_R(t)=Re^{it},\ t\in[0,\pi/2],\ \ R>0.$

5. If $z_1,\ z_2\in\mathbb{C}$ with $\text{Real}(z_1)\le 0,\ \text{Real}(z_2)\le 0,$ show that
$$|e^{z_1}\ -\ e^{z_2}|\ \le\ |z_1\ -z_2|.$$

6. Let $U\subseteq\mathbb{C}$ be open, $f,g:U\to\mathbb{C}$ be holomorphic functions with continuous derivatives and $\gamma:[a,b]\to\mathbb{C}$ be a simple, piecewise smooth curve with $\gamma^*\subset U$. If $z_0=\gamma(a),\ z_1=\gamma(b),$ show that
$$\int_\gamma f'(z)g(z)dz=f(z_1)g(z_1)-f(z_0)g(z_0)\ -\ \int_\gamma f(z)g'(z)dz.$$

7. Compute the integral $\displaystyle\int_{\gamma_r}\text{Re}z dz,$ where $\gamma_r(t)=re^{it},\ t\in[0,\ 2\pi]\ (r>0)$. By using this fact, show that the function $\text{Re}z$ does not have a primitive on any open subset of $\mathbb{C}$ containing 0.

8. (i) If $\varphi:[a,b]\to\mathbb{C}$ is differentiable $(a,b\in\mathbb{R},\ a<b),$ show that
$$\frac{d}{dt}\left[\ |\varphi(t)|^2\ \right]=2\text{Re}\left[\ \varphi'(t)\overline{\varphi(t)}\ \right],\quad \forall\, t\in[a,b].$$

(ii) Let $U\subseteq\mathbb{C}$ be open, $f:U\to\mathbb{C}$ be a holomorphic function with continuous derivative and $\gamma:[a,b]\to\mathbb{C}$ be a simple closed smooth curve with $\gamma^*\subset U$. Prove that
$$\text{Re}\left[\ \int_\gamma f'(z)\overline{f(z)}dz\ \right]=0.$$