CHAPTER

Tensors

2

As mentioned in the introduction, all laws of continuum mechanics must be formulated in terms of quantities that are independent of coordinates. It is the purpose of this chapter to introduce such mathematical entities. We begin by introducing a shorthand notation—the *indicial notation*—in Part A of this chapter, which is followed by the concept of tensors, introduced as a linear transformation in Part B. Tensor calculus is considered in Part C, and expressions for the components in cylindrical and spherical coordinates for tensors resulting from operations such as the gradient, the divergence, and the Laplacian of them are derived in Part D.

PART A: INDICIAL NOTATION

2.1 SUMMATION CONVENTION, DUMMY INDICES

Consider the sum

$$s = a_1x_1 + a_2x_2 + \ldots + a_nx_n. \tag{2.1.1}$$

We can write the preceding equation in a compact form using a summation sign:

$$s = \sum_{i=1}^{n} a_i x_i. \tag{2.1.2}$$

It is obvious that the following equations have exactly the same meaning as Eq. (2.1.2):

$$s = \sum_{j=1}^{n} a_j x_j, \quad s = \sum_{m=1}^{n} a_m x_m, \quad s = \sum_{k=1}^{n} a_k x_k. \tag{2.1.3}$$

The index i in Eq. (2.1.2), or j or m or k in Eq. (2.1.3), is a dummy index in the sense that the sum is independent of the letter used for the index. We can further simplify the writing of Eq. (2.1.1) if we adopt the following convention: Whenever an index is repeated once, it is a dummy index indicating a summation with the index running through the integral numbers 1, 2, …, n.

This convention is known as *Einstein's summation convention*. Using this convention, Eq. (2.1.1) can be written simply as:

$$s = a_i x_i \quad \text{or} \quad s = a_j x_j \quad \text{or} \quad s = a_m x_m, \qquad \text{etc.} \tag{2.1.4}$$

It is emphasized that expressions such as $a_ib_ix_i$ or $a_mb_mx_m$ are *not* defined within this convention. That is, *an index should never be repeated more than once* when the summation convention is used. Therefore, an expression of the form

$$\sum_{i=1}^{n} a_ib_ix_i,$$

must retain its summation sign.

In the following, we shall always take the number of terms n in a summation to be 3, so that, for example:

$$a_ix_i = a_1x_1 + a_2x_2 + a_3x_3, \quad a_{ii} = a_{11} + a_{22} + a_{33}.$$

The summation convention obviously can be used to express a double sum, a triple sum, and so on. For example, we can write:

$$\alpha = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j$$

concisely as

$$\alpha = a_{ij}x_ix_j. \tag{2.1.5}$$

Expanding in full, Eq. (2.1.5) gives a sum of nine terms in the right-hand side, i.e.,

$$\begin{aligned}\alpha = a_{ij}x_ix_j = {} & a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x_2x_2 + a_{23}x_2x_3 \\ & + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3x_3.\end{aligned}$$

For newcomers, it is probably better to perform the preceding expansion in two steps: first, sum over i, and then sum over j (or vice versa), i.e.,

$$a_{ij}x_ix_j = a_{1j}x_1x_j + a_{2j}x_2x_j + a_{3j}x_3x_j,$$

where

$$a_{1j}x_1x_j = a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3,$$

and so on. Similarly, the indicial notation $a_{ijk}x_ix_jx_k$ represents a triple sum of 27 terms, that is,

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a_{ijk}x_ix_jx_k = a_{ijk}x_ix_jx_k. \tag{2.1.6}$$

2.2 FREE INDICES

Consider the following system of three equations:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{aligned} \tag{2.2.1}$$

Using the summation convention, Eqs. (2.2.1) can be written as:

$$\begin{aligned} x_1' &= a_{1m}x_m, \\ x_2' &= a_{2m}x_m, \\ x_3' &= a_{3m}x_m, \end{aligned} \tag{2.2.2}$$

which can be shortened into

$$x_i' = a_{im}\, x_m, \quad i = 1, 2, 3. \tag{2.2.3}$$

An index that appears *only once* in each term of an equation such as the index i in Eq. (2.2.3) is called a *free index*. Unless stated otherwise, we agree that a free index takes on the integral number 1, 2 or 3. Thus, $x_i' = a_{im}x_m$ is shorthand for three equations, each having a sum of three terms on its right-hand side. Another simple example of a free index is the following equation defining the components of a vector **a** in terms of a dot product with each of the base vectors $\mathbf{e}_i$,

$$a_i = \mathbf{a} \cdot \mathbf{e}_i, \tag{2.2.4}$$

and clearly the vector **a** can also be expressed in terms of its components as

$$\mathbf{a} = a_i \mathbf{e}_i. \tag{2.2.5}$$

A further example is given by

$$\mathbf{e}_i' = Q_{mi}\mathbf{e}_m, \tag{2.2.6}$$

representing

$$\begin{aligned} \mathbf{e}_1' &= Q_{11}\mathbf{e}_1 + Q_{21}\mathbf{e}_2 + Q_{31}\mathbf{e}_3, \\ \mathbf{e}_2' &= Q_{12}\mathbf{e}_1 + Q_{22}\mathbf{e}_2 + Q_{32}\mathbf{e}_3, \\ \mathbf{e}_3' &= Q_{13}\mathbf{e}_1 + Q_{23}\mathbf{e}_2 + Q_{33}\mathbf{e}_3. \end{aligned} \tag{2.2.7}$$

We note that $x_j' = a_{jm}x_m$ is the same as Eq. (2.2.3) and $\mathbf{e}_j' = Q_{mj}\mathbf{e}_m$ is the same as Eq. (2.2.6). However, $a_i = b_j$ is a meaningless equation. *The free index appearing in every term of an equation must be the same*. Thus, the following equations are meaningful:

$$a_i + k_i = c_i \quad \text{or} \quad a_i + b_i c_j d_j = f_i.$$

If there are two free indices appearing in an equation such as:

$$T_{ij} = A_{im}A_{jm}, \tag{2.2.8}$$

then the equation is a shorthand for the nine equations, each with a sum of three terms on the right-hand side. In fact,

$$\begin{aligned} T_{11} &= A_{1m}A_{1m} = A_{11}A_{11} + A_{12}A_{12} + A_{13}A_{13}, \\ T_{12} &= A_{1m}A_{2m} = A_{11}A_{21} + A_{12}A_{22} + A_{13}A_{23}, \\ T_{13} &= A_{1m}A_{3m} = A_{11}A_{31} + A_{12}A_{32} + A_{13}A_{33}, \\ T_{21} &= A_{2m}A_{1m} = A_{21}A_{11} + A_{22}A_{12} + A_{23}A_{13}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ T_{33} &= A_{3m}A_{3m} = A_{31}A_{31} + A_{32}A_{32} + A_{33}A_{33}. \end{aligned}$$

2.3 THE KRONECKER DELTA

The *Kronecker delta*, denoted by δ_{ij}, is defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.3.1}$$

That is,

$$\delta_{11} = \delta_{22} = \delta_{33} = 1, \quad \delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0. \tag{2.3.2}$$

In other words, the matrix of the Kronecker delta is the identity matrix:

$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.3.3}$$

We note the following:

(a) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1,$

that is,

$$\delta_{ii} = 3. \tag{2.3.4}$$

(b) $\delta_{1m}a_m = \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 = \delta_{11}a_1 = a_1,$
$\delta_{2m}a_m = \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 = \delta_{22}a_2 = a_2,$
$\delta_{3m}a_m = \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 = \delta_{33}a_3 = a_3,$

that is,

$$\delta_{im}a_m = a_i. \tag{2.3.5}$$

(c) $\delta_{1m}T_{mj} = \delta_{11}T_{1j} + \delta_{12}T_{2j} + \delta_{13}T_{3j} = T_{1j},$
$\delta_{2m}T_{mj} = \delta_{21}T_{1j} + \delta_{22}T_{2j} + \delta_{23}T_{3j} = T_{2j},$
$\delta_{3m}T_{mj} = \delta_{31}T_{1j} + \delta_{32}T_{2j} + \delta_{33}T_{3j} = T_{3j},$

that is,

$$\delta_{im}T_{mj} = T_{ij}. \tag{2.3.6}$$

In particular,

$$\delta_{im}\delta_{mj} = \delta_{ij}, \quad \delta_{im}\delta_{mn}\delta_{nj} = \delta_{ij}, \quad \text{etc.} \tag{2.3.7}$$

(d) If $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors perpendicular to one another, then clearly,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}. \tag{2.3.8}$$

2.4 THE PERMUTATION SYMBOL

The *permutation symbol*, denoted by ε_{ijk}, is defined by:

$$\varepsilon_{ijk} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \equiv \text{according to whether } i, j, k \begin{pmatrix} \text{form an even} \\ \text{form an odd} \\ \text{do not form} \end{pmatrix} \text{permutation of } 1, 2, 3, \tag{2.4.1}$$

i.e.,

$$\begin{aligned} \varepsilon_{123} &= \varepsilon_{231} = \varepsilon_{312} = +1, \\ \varepsilon_{213} &= \varepsilon_{321} = \varepsilon_{132} = -1, \\ \varepsilon_{111} &= \varepsilon_{112} = \varepsilon_{222} = \ldots = 0. \end{aligned} \tag{2.4.2}$$

We note that

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{jik} = -\varepsilon_{kji} - \varepsilon_{ikj}. \tag{2.4.3}$$

If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a right-handed triad, then

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1, \quad \text{etc.,} \tag{2.4.4}$$

which can be written in a short form as

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k = \varepsilon_{jki}\mathbf{e}_k = \varepsilon_{kij}\mathbf{e}_k. \tag{2.4.5}$$

Now, if $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$, then, since the cross-product is distributive, we have

$$\mathbf{a} \times \mathbf{b} = (a_i\mathbf{e}_i) \times (b_j\mathbf{e}_j) = a_ib_j(\mathbf{e}_i \times \mathbf{e}_j) = a_ib_j\varepsilon_{ijk}\mathbf{e}_k. \tag{2.4.6}$$

The following useful identity can be proven (see Prob. 2.12):

$$\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{2.4.7}$$

2.5 INDICIAL NOTATION MANIPULATIONS

(a) *Substitution*: If

$$a_i = U_{im}\, b_m, \tag{i}$$

and

$$b_i = V_{im}\, c_m, \tag{ii}$$

then, in order to substitute the b_i in Eq. (ii) into the b_m in Eq. (i), we must first change the free index in Eq. (ii) from i to m and the dummy index m to some other letter—say, n—so that

$$b_m = V_{mn}\, c_n. \tag{iii}$$

Now Eqs. (i) and (iii) give

$$a_i = U_{im}V_{mn}\, c_n. \tag{iv}$$

Note that Eq. (iv) represents three equations, each having a sum of nine terms on its right-hand side.

(b) *Multiplication*: If

$$p = a_mb_m \quad \text{and} \quad q = c_md_m,$$

then

$$pq = a_mb_mc_nd_n.$$

It is important to note that $pq \neq a_mb_mc_md_m$. In fact, the right-hand side of this expression, i.e., $a_mb_mc_md_m$, is not even defined in the summation convention, and further, it is obvious that

$$pq \neq \sum_{m=1}^{3} a_mb_mc_md_m.$$

Since the dot product of vectors is distributive, therefore, if $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$, then

$$\mathbf{a}\cdot\mathbf{b} = (a_i\mathbf{e}_i)\cdot(b_j\mathbf{e}_j) = a_ib_j(\mathbf{e}_i\cdot\mathbf{e}_j).$$

In particular, if $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are unit vectors perpendicular to one another, then $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$ so that

$$\mathbf{a}\cdot\mathbf{b} = a_ib_j\delta_{ij} = a_ib_i = a_1b_1 + a_2b_2 + a_3b_3,$$

which is the familiar expression for the evaluation of the dot product in terms of the vector components.

(c) *Factoring*: If

$$T_{ij}n_j - \lambda n_i = 0,$$

then, using the Kronecker delta, we can write $n_i = \delta_{ij}n_j$, so that we have

$$T_{ij}n_j - \lambda\delta_{ij}n_j = 0.$$

Thus,

$$(T_{ij} - \lambda\delta_{ij})n_j = 0.$$

(d) *Contraction*: The operation of identifying two indices is known as a *contraction*. Contraction indicates a sum on the index. For example, T_{ii} is the contraction of T_{ij} with

$$T_{ii} = T_{11} + T_{22} + T_{33}.$$

If

$$T_{ij} = \lambda\Delta\delta_{ij} + 2\mu E_{ij},$$

then

$$T_{ii} = \lambda\Delta\delta_{ii} + 2\mu E_{ii} = 3\lambda\Delta + 2\mu E_{ii}.$$

PROBLEMS FOR PART A

2.1 Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \quad \text{and} \quad [a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

evaluate (a) S_{ii}, (b) $S_{ij}S_{ij}$, (c) $S_{ji}S_{ji}$, (d) $S_{jk}S_{kj}$, (e) a_ma_m, (f) $S_{mn}a_ma_n$, and (g) $S_{nm}a_ma_n$.

2.2 Determine which of these equations has an identical meaning with $a_i = Q_{ij}a'_j$.
(a) $a_p = Q_{pm}a'_m$, (b) $a_p = Q_{qp}a'_q$, (c) $a_m = a'_nQ_{mn}$.

2.3 Given the following matrices

$$[a_i] = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad [B_{ij}] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix},$$

demonstrate the equivalence of the subscripted equations and the corresponding matrix equations in the following two problems:

(a) $b_i = B_{ij}a_j$ and $[b] = [B][a]$ and (b) $s = B_{ij}a_ia_j$ and $s = [a]^{\mathrm{T}}[B][a]$.

2.4 Write in indicial notation the matrix equation (a) $[A]=[B][C]$, (b) $[D]=[B]^{\mathrm{T}}[C]$ and (c) $[E]=[B]^{\mathrm{T}}[C][F]$.

2.5 Write in indicial notation the equation (a) $s = A_1^2 + A_2^2 + A_3^2$ and (b) $\dfrac{\partial^2 \phi}{\partial x_1^2} + \dfrac{\partial^2 \phi}{\partial x_2^2} + \dfrac{\partial^2 \phi}{\partial x_3^2} = 0$.

2.6 Given that $S_{ij}=a_i a_j$ and $S'_{ij} = a'_i a'_j$, where $a'_i = Q_{mi} a_m$ and $a'_j = Q_{nj} a_n$, and $Q_{ik} Q_{jk} = \delta_{ij}$, show that $S'_{ii} = S_{ii}$.

2.7 Write $a_i = \dfrac{\partial v_i}{\partial t} + v_j \dfrac{\partial v_i}{\partial x_j}$ in long form.

2.8 Given that $T_{ij} = 2\mu E_{ij} + \lambda E_{kk} \delta_{ij}$, show that
(a) $T_{ij} E_{ij} = 2\mu E_{ij} E_{ij} + \lambda (E_{kk})^2$ and (b) $T_{ij} T_{ij} = 4\mu^2 E_{ij} E_{ij} + (E_{kk})^2 (4\mu\lambda + 3\lambda^2)$.

2.9 Given that $a_i = T_{ij} b_j$, and $a'_i = T'_{ij} b'_j$, where $a_i = Q_{im} a'_m$ and $T_{ij} = Q_{im} Q_{jn} T'_{mn}$,
(a) show that $Q_{im} T'_{mn} b'_n = Q_{im} Q_{jn} T'_{mn} b_j$ and (b) if $Q_{ik} Q_{im} = \delta_{km}$, then $T'_{kn}(b'_n - Q_{jn} b_j) = 0$.

2.10 Given

$$[a_i] = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad [b_i] = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix},$$

evaluate $[d_i]$, if $d_k = \varepsilon_{ijk} a_i b_j$, and show that this result is the same as $d_k = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_k$.

2.11 (a) If $\varepsilon_{ijk} T_{ij} = 0$, show that $T_{ij} = T_{ji}$, and (b) show that $\delta_{ij} \varepsilon_{ijk} = 0$.

2.12 Verify the following equation: $\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$. *Hint:* There are six cases to be considered: (1) $i = j$, (2) $i = k$, (3) $i = l$, (4) $j = k$, (5) $j = l$, and (6) $k = l$.

2.13 Use the identity $\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$ as a shortcut to obtain the following results: (a) $\varepsilon_{ilm} \varepsilon_{jlm} = 2\delta_{ij}$ and (b) $\varepsilon_{ijk} \varepsilon_{ijk} = 6$.

2.14 Use the identity $\varepsilon_{ijm} \varepsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$ to show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

2.15 Show that (a) if $T_{ij} = -T_{ji}$, then $T_{ij} a_i a_j = 0$, (b) if $T_{ij} = -T_{ji}$, and $S_{ij} = S_{ji}$, then $T_{ij} S_{ij} = 0$.

2.16 Let $T_{ij} = \dfrac{1}{2}(S_{ij} + S_{ji})$ and $R_{ij} = \dfrac{1}{2}(S_{ij} - S_{ji})$, show that $T_{ij} = T_{ji}, R_{ij} = -R_{ji}$, and $S_{ij} = T_{ij} + R_{ij}$.

2.17 Let $f(x_1, x_2, x_3)$ be a function of x_1, x_2, and x_3 and let $v_i(x_1, x_2, x_3)$ be three functions of x_1, x_2, and x_3. Express the total differential df and dv_i in indicial notation.

2.18 Let $|A_{ij}|$ denote the determinant of the matrix $[A_{ij}]$. Show that $|A_{ij}| = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$.

PART B: TENSORS

2.6 TENSOR: A LINEAR TRANSFORMATION

Let **T** be a transformation that transforms any vector into another vector. If **T** transforms **a** into **c** and **b** into **d**, we write $\mathbf{Ta} = \mathbf{c}$ and $\mathbf{Tb} = \mathbf{d}$.

If **T** has the following linear properties:

$$\mathbf{T}(\mathbf{a} + \mathbf{b}) = \mathbf{Ta} + \mathbf{Tb}, \tag{2.6.1}$$

$$\mathbf{T}(\alpha \mathbf{a}) = \alpha \mathbf{Ta}, \tag{2.6.2}$$

where **a** and **b** are two arbitrary vectors and α is an arbitrary scalar, then **T** is called a *linear transformation*. It is also called a *second-order tensor* or simply a *tensor*.* An alternative and equivalent definition of a linear transformation is given by the single linear property:

$$\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{T}\mathbf{a} + \beta\mathbf{T}\mathbf{b}, \tag{2.6.3}$$

where **a** and **b** are two arbitrary vectors and α and β are arbitrary scalars. If two tensors, **T** and **S**, transform any arbitrary vector **a** identically, these two tensors are the same, that is, if $\mathbf{Ta} = \mathbf{Sa}$ for any **a**, then $\mathbf{T} = \mathbf{S}$. We note, however, that two different tensors may transform specific vectors identically.

Example 2.6.1

Let **T** be a nonzero transformation that transforms every vector into a fixed nonzero vector **n**. Is this transformation a tensor?

Solution

Let **a** and **b** be any two vectors; then $\mathbf{Ta} = \mathbf{n}$ and $\mathbf{Tb} = \mathbf{n}$. Since $\mathbf{a} + \mathbf{b}$ is also a vector, therefore $\mathbf{T}(\mathbf{a} + \mathbf{b}) = \mathbf{n}$. Clearly $\mathbf{T}(\mathbf{a} + \mathbf{b})$ does not equal $\mathbf{Ta} + \mathbf{Tb}$. Thus, this transformation is not a linear one. In other words, it is not a tensor.

Example 2.6.2

Let **T** be a transformation that transforms every vector into a vector that is k times the original vector. Is this transformation a tensor?

Solution

Let **a** and **b** be arbitrary vectors and α and β be arbitrary scalars; then, by the definition of **T**,

$$\mathbf{Ta} = k\mathbf{a},\ \mathbf{Tb} = k\mathbf{b} \quad \text{and} \quad \mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = k(\alpha\mathbf{a} + \beta\mathbf{b}). \tag{i}$$

Clearly,

$$\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha k\mathbf{a} + \beta k\mathbf{b} = \alpha\mathbf{Ta} + \beta\mathbf{Tb}. \tag{ii}$$

Therefore, **T** is a linear transformation. In other words, it is a tensor. If $k = 0$, then the tensor transforms all vectors into a zero vector (null vector). This tensor is the *zero tensor* or *null tensor* and is symbolized by the boldface **0**.

Example 2.6.3

Consider a transformation **T** that transforms every vector into its mirror image with respect to a fixed plane. Is **T** a tensor?

Solution

Consider a parallelogram in space with its sides representing vectors **a** and **b** and its diagonal the vector sum of **a** and **b**. Since the parallelogram remains a parallelogram after the reflection, the diagonal (the resultant vector)

*Scalars and vectors are sometimes called the *zeroth order tensor* and the *first-order tensor*, respectively. Even though they can also be defined algebraically, in terms of certain operational rules, we choose not to do so. The geometrical concept of scalars and vectors, with which we assume readers are familiar, is quite sufficient for our purpose.

of the reflected parallelogram is clearly both **T**(**a** + **b**) (the reflected **a** + **b**) and **Ta** + **Tb** (the sum of the reflected **a** and the reflected **b**). That is, **T**(**a** + **b**) = **Ta** + **Tb**. Also, for an arbitrary scalar α, the reflection of α**a** is obviously the same as α times the reflection of **a**, that is, **T**(α**a**) = α(**Ta**), because both vectors have the same magnitude given by α times the magnitude of **a** and in the same direction. Thus, **T** is a tensor.

Example 2.6.4

When a rigid body undergoes a rotation about some axis **n**, vectors drawn in the rigid body in general change their directions. That is, the rotation transforms vectors drawn in the rigid body into other vectors. Denote this transformation by **R**. Is **R** a tensor?

Solution

Consider a parallelogram embedded in the rigid body with its sides representing vectors **a** and **b** and its diagonal representing the resultant (**a** + **b**). Since the parallelogram remains a parallelogram after a rotation about any axis, the diagonal (the resultant vector) of the rotated parallelogram is clearly both **R**(**a** + **b**) (the rotated **a** + **b**) and **Ra** + **Rb** (the sum of the rotated **a** and the rotated **b**). That is, **R(a** + **b)** = **Ra** + **Rb**. A similar argument as that used in the previous example leads to **R**(α**a**) = α(**Ra**). Thus, **R** is a tensor.

Example 2.6.5

Let **T** be a tensor that transforms the specific vectors **a** and **b** as follows:

$$\mathbf{Ta} = \mathbf{a} + 2\mathbf{b},$$
$$\mathbf{Tb} = \mathbf{a} - \mathbf{b}.$$

Given a vector **c** = 2**a** + **b**, find **Tc**.

Solution

Using the linearity property of tensors, we have

$$\mathbf{Tc} = \mathbf{T}(2\mathbf{a} + \mathbf{b}) = 2\mathbf{Ta} + \mathbf{Tb} = 2(\mathbf{a} + 2\mathbf{b}) + (\mathbf{a} - \mathbf{b}) = 3\mathbf{a} + 3\mathbf{b}.$$

2.7 COMPONENTS OF A TENSOR

The components of a vector depend on the base vectors used to describe the components. This will also be true for tensors.

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be unit vectors in the direction of the x_1-, x_2-, x_3-, respectively, of a rectangular Cartesian coordinate system. Under a transformation **T**, these vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ become $\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{Te}_3$. Each of these $\mathbf{Te}_i$, being a vector, can be written as:

$$\begin{aligned} \mathbf{Te}_1 &= T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3, \\ \mathbf{Te}_2 &= T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3, \\ \mathbf{Te}_3 &= T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3, \end{aligned} \tag{2.7.1}$$

or

$$\mathbf{Te}_i = T_{ji}\mathbf{e}_j. \tag{2.7.2}$$

The components T_{ij} in the preceding equations are defined as the components of the tensor **T**. These components can be put in a matrix as follows:

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}. \tag{2.7.3}$$

This matrix is called the *matrix of the tensor* **T** with respect to the set of base vectors $\{\mathbf{e}_i\}$. We note that, because of the way we have chosen to denote the components of transformation of the base vectors, the elements of the first column in the matrix are components of the vector $\mathbf{Te}_1$, those in the second column are the components of the vector $\mathbf{Te}_2$, and those in the third column are the components of $\mathbf{Te}_3$.

Example 2.7.1

Obtain the matrix for the tensor **T** that transforms the base vectors as follows:

$$\begin{aligned} \mathbf{Te}_1 &= 4\mathbf{e}_1 + \mathbf{e}_2, \\ \mathbf{Te}_2 &= 2\mathbf{e}_1 + 3\mathbf{e}_3, \\ \mathbf{Te}_3 &= -\mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3. \end{aligned} \tag{i}$$

Solution

By Eq. (2.7.1),

$$[\mathbf{T}] = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}. \tag{ii}$$

Example 2.7.2

Let **T** transform every vector into its mirror image with respect to a fixed plane; if $\mathbf{e}_1$ is normal to the reflection plane ($\mathbf{e}_2$ and $\mathbf{e}_3$ are parallel to this plane), find a matrix of **T**.

Solution

Since the normal to the reflection plane is transformed into its negative and vectors parallel to the plane are not altered, we have

$$\mathbf{Te}_1 = -\mathbf{e}_1, \quad \mathbf{Te}_2 = \mathbf{e}_2, \quad \mathbf{Te}_3 = \mathbf{e}_3$$

which corresponds to

$$[\mathbf{T}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i}.$$

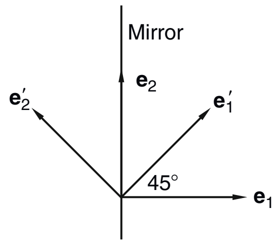

FIGURE 2.7-1

We note that this is only one of the infinitely many matrices of the tensor **T**; each depends on a particular choice of base vectors. In the preceding matrix, the choice of $\mathbf{e}_i$ is indicated at the bottom-right corner of the matrix. If we choose $\mathbf{e}_1'$ and $\mathbf{e}_2'$ to be on a plane perpendicular to the mirror, with each making 45° with the mirror, as shown in Figure 2.7-1, and $\mathbf{e}_3'$ pointing straight out from the paper, then we have

$$\mathbf{T}\mathbf{e}_1' = \mathbf{e}_2', \quad \mathbf{T}\mathbf{e}_2' = \mathbf{e}_1', \quad \mathbf{T}\mathbf{e}_3' = \mathbf{e}_3'.$$

Thus, with respect to $\{\mathbf{e}_i'\}$, the matrix of the tensor is

$$[\mathbf{T}]' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i'}.$$

Throughout this book, we denote the matrix of a tensor **T** with respect to the basis $\{\mathbf{e}_i\}$ by either $[\mathbf{T}]$ or $[T_{ij}]$ and with respect to the basis $\{\mathbf{e}_i'\}$ by either $[\mathbf{T}]'$ or $[T_{ij}']$. The last two matrices should not be confused with $[\mathbf{T}']$, which represents the matrix of the tensor $\mathbf{T}'$ with respect to the basis $\{\mathbf{e}_i\}$, not the matrix of **T** with respect to the primed basis $\{\mathbf{e}_i'\}$.

Example 2.7.3

Let **R** correspond to a right-hand rotation of a rigid body about the x_3-axis by an angle θ (Figure 2.7-2). Find a matrix of **R**.

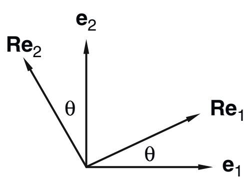

FIGURE 2.7-2

Solution

From Figure 2.7-2, it is clear that

$$\mathbf{R}\mathbf{e}_1 = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2,$$
$$\mathbf{R}\mathbf{e}_2 = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2,,$$
$$\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3.$$

which corresponds to

$$[\mathbf{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i}.$$

Example 2.7.4

Obtain the matrix for the tensor **T**, which transforms the base vectors as follows:

$$\mathbf{Te}_1 = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3,$$
$$\mathbf{Te}_2 = 4\mathbf{e}_1 + 5\mathbf{e}_2 + 6\mathbf{e}_3,$$
$$\mathbf{Te}_3 = 7\mathbf{e}_1 + 8\mathbf{e}_2 + 9\mathbf{e}_3.$$

Solution

By inspection,

$$[\mathbf{T}] = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

This example emphasizes again the convention we use to write the matrix of a tensor: The components of $\mathbf{Te}_1$ fill the first column, the components of $\mathbf{Te}_2$ fill the second column, and so on. The reason for this choice of convention will become obvious in the next section.

Since $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0$ (because they are mutually perpendicular), it can be easily verified from Eq. (2.7.1) that

$$\begin{aligned} T_{11} &= \mathbf{e}_1 \cdot \mathbf{Te}_1, \quad & T_{12} &= \mathbf{e}_1 \cdot \mathbf{Te}_2, \quad & T_{13} &= \mathbf{e}_1 \cdot \mathbf{Te}_3, \\ T_{21} &= \mathbf{e}_2 \cdot \mathbf{Te}_1, \quad & T_{22} &= \mathbf{e}_2 \cdot \mathbf{Te}_2, \quad & T_{23} &= \mathbf{e}_2 \cdot \mathbf{Te}_3, \\ T_{31} &= \mathbf{e}_3 \cdot \mathbf{Te}_1, \quad & T_{32} &= \mathbf{e}_3 \cdot \mathbf{Te}_2, \quad & T_{33} &= \mathbf{e}_3 \cdot \mathbf{Te}_3, \end{aligned} \tag{2.7.4}$$

or

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{Te}_j. \tag{2.7.5}$$

These equations are totally equivalent to Eq. (2.7.1) [or Eq. (2.7.2)] and can also be regarded as the definition of the components of a tensor **T**. They are often more convenient to use than Eq. (2.7.2).

We note again that the components of a tensor depend on the coordinate systems through the set of base vectors. Thus,

$$T'_{ij} = \mathbf{e}'_i \cdot \mathbf{Te}'_j, \tag{2.7.6}$$

where T'_{ij} are the components of the same tensor **T** with respect to the base vectors $\{\mathbf{e}'_i\}$. It is important to note that vectors and tensors are independent of coordinate systems, but *their components* are dependent on the coordinate systems.

2.8 COMPONENTS OF A TRANSFORMED VECTOR

Given the vector **a** and the tensor **T**, which transforms **a** into **b** (i.e., $\mathbf{b} = \mathbf{Ta}$), we wish to compute the components of **b** from the components of **a** and the components of **T**. Let the components of **a** with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be (a_1, a_2, a_3), that is,

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3, \tag{2.8.1}$$

then

$$\mathbf{b} = \mathbf{Ta} = \mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1\mathbf{Te}_1 + a_2\mathbf{Te}_2 + a_3\mathbf{Te}_3,$$

thus,

$$\begin{aligned} b_1 &= \mathbf{b}\cdot\mathbf{e}_1 = \mathbf{e}_1\cdot\mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1(\mathbf{e}_1\cdot\mathbf{Te}_1) + a_2(\mathbf{e}_1\cdot\mathbf{Te}_2) + a_3(\mathbf{e}_1\cdot\mathbf{Te}_3),\\ b_2 &= \mathbf{b}\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1(\mathbf{e}_2\cdot\mathbf{Te}_1) + a_2(\mathbf{e}_2\cdot\mathbf{Te}_2) + a_3(\mathbf{e}_2\cdot\mathbf{Te}_3),\\ b_3 &= \mathbf{b}\cdot\mathbf{e}_3 = \mathbf{e}_3\cdot\mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1(\mathbf{e}_3\cdot\mathbf{Te}_1) + a_2(\mathbf{e}_3\cdot\mathbf{Te}_2) + a_3(\mathbf{e}_3\cdot\mathbf{Te}_3). \end{aligned}$$

By Eqs. (2.7.4), we have

$$\begin{aligned} b_1 &= T_{11}a_1 + T_{12}a_2 + T_{13}a_3,\\ b_2 &= T_{21}a_1 + T_{22}a_2 + T_{23}a_3,\\ b_3 &= T_{31}a_1 + T_{32}a_2 + T_{33}a_3. \end{aligned} \tag{2.8.2}$$

We can write the preceding three equations in matrix form as:

$$\begin{bmatrix} b_1\\ b_2\\ b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_1\\ a_2\\ a_3 \end{bmatrix}, \tag{2.8.3}$$

or

$$[\mathbf{b}] = [\mathbf{T}][\mathbf{a}]. \tag{2.8.4}$$

We can also derive Eq. (2.8.2) using indicial notations as follows: From $\mathbf{a} = a_i\mathbf{e}_i$, we get $\mathbf{Ta} = \mathbf{T}(a_i\mathbf{e}_i) = a_i\mathbf{Te}_i$. Since $\mathbf{Te}_i = T_{ji}\mathbf{e}_j$ [Eq. (2.7.2)], $\mathbf{b} = \mathbf{Ta} = a_iT_{ji}\mathbf{e}_j$ so that

$$b_m = \mathbf{b}\cdot\mathbf{e}_m = a_iT_{ji}\mathbf{e}_j\cdot\mathbf{e}_m = a_iT_{ji}\delta_{jm} = a_iT_{mi},$$

that is,

$$b_m = a_iT_{mi} = T_{mi}a_i. \tag{2.8.5}$$

Eq. (2.8.5) is nothing but Eq. (2.8.2) in indicial notation.

We see that for the tensorial equation $\mathbf{b} = \mathbf{Ta}$, there corresponds a matrix equation of exactly the same form, that is, $[\mathbf{b}] = [\mathbf{T}][\mathbf{a}]$. This is the reason we adopted the convention that $\mathbf{Te}_i = T_{ji}\mathbf{e}_j$ (i.e., $\mathbf{Te}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3$, etc.). If we had adopted the convention that $\mathbf{Te}_i = T_{ij}\mathbf{e}_j$ (i.e., $\mathbf{Te}_1 = T_{11}\mathbf{e}_1 + T_{12}\mathbf{e}_2 + T_{13}\mathbf{e}_3$, etc.), then we would have obtained $[\mathbf{b}] = [\mathbf{T}]^{\mathrm{T}}[\mathbf{a}]$ for the tensorial equation $\mathbf{b} = \mathbf{Ta}$, which would not be as natural.

Example 2.8.1

Given that a tensor $\mathbf{T}$ transforms the base vectors as follows:

$$\begin{aligned} \mathbf{Te}_1 &= 2\mathbf{e}_1 - 6\mathbf{e}_2 + 4\mathbf{e}_3,\\ \mathbf{Te}_2 &= 3\mathbf{e}_1 + 4\mathbf{e}_2 - 1\mathbf{e}_3,\\ \mathbf{Te}_3 &= -2\mathbf{e}_1 + 1\mathbf{e}_2 + 2\mathbf{e}_3. \end{aligned}$$

how does this tensor transform the vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$?

Solution
Use the matrix equation

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -2 \\ -6 & 4 & 1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix},$$

we obtain $\mathbf{b} = 2\mathbf{e}_1 + 5\mathbf{e}_2 + 8\mathbf{e}_3$.

2.9 SUM OF TENSORS

Let **T** and **S** be two tensors. The sum of **T** and **S**, denoted by **T** + **S**, is defined by

$$(\mathbf{T} + \mathbf{S})\mathbf{a} = \mathbf{Ta} + \mathbf{Sa} \tag{2.9.1}$$

for any vector **a**. It is easily seen that **T** + **S**, so defined, is indeed a tensor. To find the components of **T** + **S**, let

$$\mathbf{W} = \mathbf{T} + \mathbf{S}. \tag{2.9.2}$$

The components of **W** are [see Eqs. (2.7.5)]

$$W_{ij} = \mathbf{e}_i \cdot (\mathbf{T} + \mathbf{S})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{Te}_j + \mathbf{e}_i \cdot \mathbf{Se}_j,$$

that is,

$$W_{ij} = T_{ij} + S_{ij}. \tag{2.9.3}$$

In matrix notation, we have

$$[\mathbf{W}] = [\mathbf{T}] + [\mathbf{S}], \tag{2.9.4}$$

and that the tensor sum is consistent with the matrix sum.

2.10 PRODUCT OF TWO TENSORS

Let **T** and **S** be two tensors and **a** be an arbitrary vector. Then **TS** and **ST** are defined to be the transformations (easily seen to be tensors) such that

$$(\mathbf{TS})\mathbf{a} = \mathbf{T}(\mathbf{Sa}), \tag{2.10.1}$$

and

$$(\mathbf{ST})\mathbf{a} = \mathbf{S}(\mathbf{Ta}). \tag{2.10.2}$$

The components of **TS** are

$$(\mathbf{TS})_{ij} = \mathbf{e}_i \cdot (\mathbf{TS})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{T}(\mathbf{Se}_j) = \mathbf{e}_i \cdot \mathbf{T}S_{mj}\mathbf{e}_m = S_{mj}\mathbf{e}_i \cdot \mathbf{Te}_m = S_{mj}T_{im}, \tag{2.10.3}$$

that is,

$$(\mathbf{TS})_{ij} = T_{im}S_{mj}. \tag{2.10.4}$$

Similarly,

$$(\mathbf{ST})_{ij} = S_{im}T_{mj}. \tag{2.10.5}$$

Eq. (2.10.4) is equivalent to the matrix equation:

$$[\mathbf{TS}] = [\mathbf{T}][\mathbf{S}], \tag{2.10.6}$$

whereas Eq. (2.10.5) is equivalent to the matrix equation:

$$[\mathbf{ST}] = [\mathbf{S}][\mathbf{T}]. \tag{2.10.7}$$

The two products are, in general, different. Thus, it is clear that in general $\mathbf{TS} \neq \mathbf{ST}$. That is, in general, the tensor product is not commutative.

If **T, S**, and **V** are three tensors, then, by repeatedly using the definition (2.10.1), we have

$$(\mathbf{T}(\mathbf{SV}))\mathbf{a} \equiv \mathbf{T}((\mathbf{SV})\mathbf{a}) \equiv \mathbf{T}(\mathbf{S}(\mathbf{Va})) \quad \text{and} \quad (\mathbf{TS})(\mathbf{Va}) \equiv \mathbf{T}(\mathbf{S}(\mathbf{Va})), \tag{2.10.8}$$

that is,

$$\mathbf{T}(\mathbf{SV}) = (\mathbf{TS})\mathbf{V} = \mathbf{TSV}. \tag{2.10.9}$$

Thus, the tensor product is associative. It is, therefore, natural to define the integral positive powers of a tensor by these simple products, so that

$$\mathbf{T}^2 = \mathbf{TT}, \quad \mathbf{T}^3 = \mathbf{TTT}, \ldots \tag{2.10.10}$$

Example 2.10.1

(a) Let **R** correspond to a 90° right-hand rigid body rotation about the x_3-axis. Find the matrix of **R**.

(b) Let **S** correspond to a 90° right-hand rigid body rotation about the x_1-axis. Find the matrix of **S**.

(c) Find the matrix of the tensor that corresponds to the rotation **R**, followed by **S**.

(d) Find the matrix of the tensor that corresponds to the rotation **S**, followed by **R**.

(e) Consider a point *P* whose initial coordinates are (1,1,0). Find the new position of this point after the rotations of part (c). Also find the new position of this point after the rotations of part (d).

Solution

(a) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a set of right-handed unit base vector with $\mathbf{e}_3$ along the axis of rotation of the rigid body. Then,

$$\mathbf{Re}_1 = \mathbf{e}_2, \quad \mathbf{Re}_2 = -\mathbf{e}_1, \quad \mathbf{Re}_3 = \mathbf{e}_3,$$

that is,

$$[\mathbf{R}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) In a manner similar to (a), the transformation of the base vectors is given by:

$$\mathbf{Se}_1 = \mathbf{e}_1, \quad \mathbf{Se}_2 = \mathbf{e}_3, \quad \mathbf{Se}_3 = -\mathbf{e}_2,$$

that is,

$$[\mathbf{S}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(c) Since $\mathbf{S}(\mathbf{Ra}) = (\mathbf{SR})\mathbf{a}$, the resultant rotation is given by the single transformation **SR** whose components are given by the matrix:

$$[\mathbf{SR}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

(d) In a manner similar to (c), the resultant rotation is given by the single transformation **RS** whose components are given by the matrix:

$$[\mathbf{RS}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(e) Let $\mathbf{r}$ be the initial position of the material point P. Let $\mathbf{r}^*$ and $\mathbf{r}^{**}$ be the rotated position of P after the rotations of part (c) and part (d), respectively. Then

$$[\mathbf{r}^*] = [\mathbf{SR}][\mathbf{r}] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

that is,

$$\mathbf{r}^* = -\mathbf{e}_1 + \mathbf{e}_3,$$

and

$$[\mathbf{r}^{**}] = [\mathbf{RS}][\mathbf{r}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

that is,

$$\mathbf{r}^{**} = \mathbf{e}_2 + \mathbf{e}_3.$$

This example further illustrates that the order of rotations is significant.

2.11 TRANSPOSE OF A TENSOR

The transpose of a tensor $\mathbf{T}$, denoted by $\mathbf{T}^{\mathrm{T}}$, is defined to be the tensor that satisfies the following identity for all vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} \cdot \mathbf{Tb} = \mathbf{b} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{a}. \tag{2.11.1}$$

It can be easily seen that $\mathbf{T}^{\mathrm{T}}$ is a tensor (see Prob. 2.34). From the preceding definition, we have

$$\mathbf{e}_j \cdot \mathbf{Te}_i = \mathbf{e}_i \cdot \mathbf{T}^{\mathrm{T}}\mathbf{e}_j. \tag{2.11.2}$$

Thus,

$$T_{ji} = T_{ij}^{\mathrm{T}}, \tag{2.11.3}$$

or

$$[\mathbf{T}]^{\mathrm{T}} = [\mathbf{T}^{\mathrm{T}}], \tag{2.11.4}$$

that is, the matrix of $\mathbf{T}^{\mathrm{T}}$ is the transpose of the matrix $\mathbf{T}$. We also note that by Eq. (2.11.1), we have

$$\mathbf{a} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{b} = \mathbf{b} \cdot (\mathbf{T}^{\mathrm{T}})^{\mathrm{T}}\mathbf{a}. \tag{2.11.5}$$

Thus, $\mathbf{b} \cdot \mathbf{Ta} = \mathbf{b} \cdot (\mathbf{T}^{\mathrm{T}})^{\mathrm{T}}\mathbf{a}$ for any $\mathbf{a}$ and $\mathbf{b}$, so that

$$(\mathbf{T}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{T}. \tag{2.11.6}$$

It can be easily established that (see Prob. 2.34)

$$(\mathbf{TS})^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}. \tag{2.11.7}$$

That is, the transpose of a product of the tensors is equal to the product of transposed tensors in reverse order, which is consistent with the equivalent matrix identity. More generally,

$$(\mathbf{ABC} \ldots \mathbf{D})^{\mathrm{T}} = \mathbf{D}^{\mathrm{T}} \ldots \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}. \tag{2.11.8}$$

2.12 DYADIC PRODUCT OF VECTORS

The dyadic product of vectors $\mathbf{a}$ and $\mathbf{b}$, denoted* by $\mathbf{ab}$, is defined to be the transformation that transforms any vector $\mathbf{c}$ according to the rule:

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}). \tag{2.12.1}$$

Now, for any vectors $\mathbf{c}, \mathbf{d}$, and any scalars α and β, we have, from the preceding rule,

$$\begin{aligned}(\mathbf{ab})(\alpha\mathbf{c} + \beta\mathbf{d}) &= \mathbf{a}(\mathbf{b} \cdot (\alpha\mathbf{c} + \beta\mathbf{d})) = \mathbf{a}((\alpha\mathbf{b} \cdot \mathbf{c}) + (\beta\mathbf{b} \cdot \mathbf{d})) = \alpha\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \beta\mathbf{a}(\mathbf{b} \cdot \mathbf{d}) \\ &= \alpha(\mathbf{ab})\mathbf{c} + \beta(\mathbf{ab})\mathbf{d}.\end{aligned} \tag{2.12.2}$$

Thus, the dyadic product $\mathbf{ab}$ is a linear transformation.

Let $\mathbf{W} = \mathbf{ab}$, then the components of $\mathbf{W}$ are:

$$W_{ij} = \mathbf{e}_i \cdot \mathbf{W}\mathbf{e}_j = \mathbf{e}_i \cdot (\mathbf{ab})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{a}(\mathbf{b} \cdot \mathbf{e}_j) = a_i b_j, \tag{2.12.3}$$

that is,

$$W_{ij} = a_i b_j, \tag{2.12.4}$$

or

$$[\mathbf{W}] = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \quad b_2 \quad b_3]. \tag{2.12.5}$$

In particular, the dyadic products of the base vectors $\mathbf{e}_i$ are:

$$[\mathbf{e}_1\mathbf{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{e}_1\mathbf{e}_2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots. \tag{2.12.6}$$

*Some authors write $\mathbf{a} \otimes \mathbf{b}$ for $\mathbf{ab}$. Also, some authors write $(\mathbf{ab}) \cdot \mathbf{c}$ for $(\mathbf{ab})\mathbf{c}$ and $\mathbf{c} \cdot (\mathbf{ab})$ for $(\mathbf{ab})^{\mathrm{T}}\mathbf{c}$.

Thus, it is clear that any tensor $\mathbf{T}$ can be expressed as:

$$\mathbf{T} = T_{11}\mathbf{e}_1\mathbf{e}_1 + T_{12}\mathbf{e}_1\mathbf{e}_2 + T_{13}\mathbf{e}_1\mathbf{e}_3 + T_{21}\mathbf{e}_2\mathbf{e}_1 + \ldots = T_{ij}\mathbf{e}_i\mathbf{e}_j. \quad (2.12.7)$$

2.13 TRACE OF A TENSOR

The *trace* of a tensor is a scalar that obeys the following rules: For any tensor $\mathbf{T}$ and $\mathbf{S}$ and any vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\begin{aligned} \mathrm{tr}(\mathbf{T}+\mathbf{S}) &= \mathrm{tr}\,\mathbf{T} + \mathrm{tr}\,\mathbf{S}, \\ \mathrm{tr}(\alpha\mathbf{T}) &= \alpha\mathrm{tr}\,\mathbf{T}, \\ \mathrm{tr}(\mathbf{ab}) &= \mathbf{a}\cdot\mathbf{b}. \end{aligned} \quad (2.13.1)$$

In terms of tensor components, using Eq. (2.12.7),

$$\mathrm{tr}\,\mathbf{T} = \mathrm{tr}(T_{ij}\mathbf{e}_i\mathbf{e}_j) = T_{ij}\mathrm{tr}(\mathbf{e}_i\mathbf{e}_j) = T_{ij}\mathbf{e}_i\cdot\mathbf{e}_j = T_{ij}\delta_{ij} = T_{ii}. \quad (2.13.2)$$

That is,

$$\mathrm{tr}\,\mathbf{T} = T_{11} + T_{22} + T_{33} = \text{sum of diagonal elements}. \quad (2.13.3)$$

It is, therefore, obvious that

$$\mathrm{tr}\,\mathbf{T}^{\mathrm{T}} = \mathrm{tr}\,\mathbf{T}. \quad (2.13.4)$$

Example 2.13.1

Show that for any second-order tensor $\mathbf{A}$ and $\mathbf{B}$

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}). \quad (2.13.5)$$

Solution

Let $\mathbf{C} = \mathbf{AB}$, then $C_{ij} = A_{im}B_{mj}$, so that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}\,\mathbf{C} = C_{ii} = A_{im}B_{mi}$.

Let $\mathbf{D} = \mathbf{BA}$, then $D_{ij} = B_{im}A_{mj}$, so that $\mathrm{tr}(\mathbf{BA}) = \mathrm{tr}\,\mathbf{D} = D_{ii} = B_{im}A_{mi}$. But $B_{im}A_{mi} = B_{mi}A_{im}$ (change of dummy indices); therefore, we have the desired result

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}).$$

2.14 IDENTITY TENSOR AND TENSOR INVERSE

The linear transformation that transforms every vector into itself is called an *identity tensor*. Denoting this special tensor by $\mathbf{I}$, we have for any vector $\mathbf{a}$,

$$\mathbf{Ia} = \mathbf{a}. \quad (2.14.1)$$

In particular,

$$\mathbf{Ie}_1 = \mathbf{e}_1, \quad \mathbf{Ie}_2 = \mathbf{e}_2, \quad \mathbf{Ie}_3 = \mathbf{e}_3. \quad (2.14.2)$$

Thus the (Cartesian) components of the identity tensor are:

$$I_{ij} = \mathbf{e}_i \cdot \mathbf{I}\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{2.14.3}$$

that is,

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.14.4}$$

It is obvious that the identity matrix is the matrix of $\mathbf{I}$ for *all rectangular Cartesian coordinates* and that $\mathbf{TI} = \mathbf{IT} = \mathbf{T}$ for any tensor $\mathbf{T}$. We also note that if $\mathbf{Ta} = \mathbf{a}$ for any arbitrary $\mathbf{a}$, then $\mathbf{T} = \mathbf{I}$.

Example 2.14.1

Write the tensor $\mathbf{T}$, defined by the equation $\mathbf{Ta} = \alpha\mathbf{a}$, where α is a constant and $\mathbf{a}$ is arbitrary, in terms of the identity tensor, and find its components.

Solution

Using Eq. (2.14.1), we can write $\alpha\mathbf{a}$ as $\alpha\mathbf{Ia}$, so that

$$\mathbf{Ta} = \alpha\mathbf{a} = \alpha\mathbf{Ia}.$$

Since $\mathbf{a}$ is arbitrary, therefore,

$$\mathbf{T} = \alpha\mathbf{I}.$$

The components of this tensor are clearly $T_{ij} = \alpha\delta_{ij}$.

Given a tensor $\mathbf{T}$, if a tensor $\mathbf{S}$ exists such that

$$\mathbf{ST} = \mathbf{I}, \tag{2.14.5}$$

then we call $\mathbf{S}$ the inverse of $\mathbf{T}$ and write

$$\mathbf{S} = \mathbf{T}^{-1}. \tag{2.14.6}$$

To find the components of the inverse of a tensor $\mathbf{T}$ is to find the inverse of the matrix of $\mathbf{T}$. From the study of matrices, we know that the inverse exists if and only if $\det \mathbf{T} \neq 0$ (that is, $\mathbf{T}$ is nonsingular) and in this case,

$$[\mathbf{T}]^{-1}[\mathbf{T}] = [\mathbf{T}][\mathbf{T}]^{-1} = [\mathbf{I}]. \tag{2.14.7}$$

Thus, the inverse of a tensor satisfies the following relation:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{T}\mathbf{T}^{-1} = \mathbf{I}. \tag{2.14.8}$$

It can be shown (see Prob. 2.35) that for the tensor inverse, the following relations are satisfied:

$$(\mathbf{T}^{\mathrm{T}})^{-1} = (\mathbf{T}^{-1})^{\mathrm{T}}, \tag{2.14.9}$$

and

$$(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}. \tag{2.14.10}$$

We note that if the inverse exists, we have the reciprocal relations that

$$\mathbf{Ta} = \mathbf{b} \quad \text{and} \quad \mathbf{a} = \mathbf{T}^{-1}\mathbf{b}. \tag{2.14.11}$$

This indicates that when a tensor is invertible, there is a one-to-one mapping of vectors **a** and **b**. On the other hand, if a tensor **T** does not have an inverse, then, for a given **b**, there are in general more than one **a** that transform into **b**. This fact is illustrated in the following example.

Example 2.14.2

Consider the tensor **T** = **cd** (the dyadic product of **c** and **d**).

(a) Obtain the determinant of **T**.
(b) Show that if **Ta** = **b**, then **T**(**a** + **h**) = **b**, where **h** is any vector perpendicular to the vector **d**.

Solution

(a) $[\mathbf{T}] = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} [\, d_1 \quad d_2 \quad d_3 \,] = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 \\ c_2d_1 & c_2d_2 & c_2d_3 \\ c_3d_1 & c_3d_2 & c_3d_3 \end{bmatrix}$ and $\det[\mathbf{T}] = c_1c_2c_3d_1d_2d_3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$

That is, **T** is a singular tensor, for which an inverse does not exist.

(b) $\mathbf{T}(\mathbf{a}+\mathbf{h}) = (\mathbf{cd})(\mathbf{a}+\mathbf{h}) = \mathbf{c}(\mathbf{d}\cdot\mathbf{a}) + \mathbf{c}(\mathbf{d}\cdot\mathbf{h})$. Now $\mathbf{d}\cdot\mathbf{h} = 0$ (**h** is perpendicular to **d**); therefore,

$$\mathbf{T}(\mathbf{a}+\mathbf{h}) = \mathbf{c}(\mathbf{d}\cdot\mathbf{a}) = (\mathbf{cd})\mathbf{a} = \mathbf{Ta} = \mathbf{b}.$$

That is, all vectors **a** + **h** transform into the vector **b**, and it is not a one-to-one transformation.

2.15 ORTHOGONAL TENSORS

An *orthogonal tensor* is a linear transformation under which the transformed vectors preserve their lengths and angles. Let **Q** denote an orthogonal tensor; then by definition, $|\mathbf{Qa}| = |\mathbf{a}|$, $|\mathbf{Qb}| = |\mathbf{b}|$, and $\cos(\mathbf{a},\mathbf{b}) = \cos(\mathbf{Qa}, \mathbf{Qb})$. Therefore,

$$\mathbf{Qa}\cdot\mathbf{Qb} = \mathbf{a}\cdot\mathbf{b} \tag{2.15.1}$$

for any vectors **a** and **b**.

Since by the definition of transpose, Eq. (2.11.1), $(\mathbf{Qa})\cdot(\mathbf{Qb}) = \mathbf{b}\cdot\mathbf{Q}^{\mathrm{T}}(\mathbf{Qa})$, thus

$$\mathbf{b}\cdot\mathbf{a} = \mathbf{b}\cdot(\mathbf{Q}^{\mathrm{T}}\mathbf{Q})\mathbf{a} \quad \text{or} \quad \mathbf{b}\cdot\mathbf{Ia} = \mathbf{b}\cdot\mathbf{Q}^{\mathrm{T}}\mathbf{Qa}.$$

Since **a** and **b** are arbitrary, it follows that

$$\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}. \tag{2.15.2}$$

This means that for an orthogonal tensor, the inverse is simply the transpose,

$$\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}. \tag{2.15.3}$$

Thus [see Eq. (2.14.8)],

$$\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}. \tag{2.15.4}$$

In matrix notation, Eq. (2.15.4) takes the form:

$$[\mathbf{Q}]^{\mathrm{T}}[\mathbf{Q}] = [\mathbf{Q}][\mathbf{Q}]^{\mathrm{T}} = [\mathbf{I}], \tag{2.15.5}$$

and in subscript notation, we have

$$Q_{mi}Q_{mj} = Q_{im}Q_{jm} = \delta_{ij}. \tag{2.15.6}$$

Example 2.15.1

The tensor given in Example 2.7.2, being a reflection, is obviously an orthogonal tensor. Verify that $[\mathbf{T}][\mathbf{T}]^{\mathrm{T}} = [\mathbf{I}]$ for the $[\mathbf{T}]$ in that example. Also, find the determinant of $[\mathbf{T}]$.

Solution

Evaluating the matrix product:

$$[\mathbf{T}][\mathbf{T}]^{\mathrm{T}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant of $\mathbf{T}$ is

$$|\mathbf{T}| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1.$$

Example 2.15.2

The tensor given in Example 2.7.3, being a rigid body rotation, is obviously an orthogonal tensor. Verify that $[\mathbf{R}][\mathbf{R}]^{\mathrm{T}} = [\mathbf{I}]$ for the $[\mathbf{R}]$ in that example. Also find the determinant of $[\mathbf{R}]$.

Solution

$$[\mathbf{R}][\mathbf{R}]^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\det[\mathbf{R}] = |\mathbf{R}| = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

The determinant of the matrix of any orthogonal tensor $\mathbf{Q}$ is easily shown to be equal to either $+1$ or -1. In fact, since

$$[\mathbf{Q}][\mathbf{Q}]^{\mathrm{T}} = [\mathbf{I}],$$

therefore,

$$|[\mathbf{Q}][\mathbf{Q}]^{\mathrm{T}}| = |\mathbf{Q}||\mathbf{Q}^{\mathrm{T}}| = |\mathbf{I}|.$$

Now $|\mathbf{Q}| = |\mathbf{Q}^T|$ and $|\mathbf{I}| = 1$, therefore, $|\mathbf{Q}|^2 = 1$, thus

$$|\mathbf{Q}| = \pm 1. \tag{2.15.7}$$

From the previous examples, we can see that for a rotation tensor the determinant is $+1$, whereas for a reflection tensor, it is -1.

2.16 TRANSFORMATION MATRIX BETWEEN TWO RECTANGULAR CARTESIAN COORDINATE SYSTEMS

Suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ are unit vectors corresponding to two rectangular Cartesian coordinate systems (see Figure 2.16-1). It is clear that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ can be made to coincide with $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ through either a rigid body rotation (if both bases are same-handed) or a rotation followed by a reflection (if different-handed). That is, $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$ are related by an orthogonal tensor $\mathbf{Q}$ through the equations below.

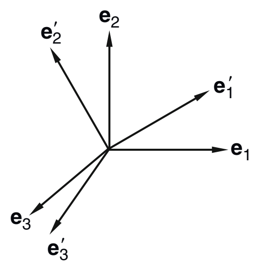

FIGURE 2.16-1

$$\mathbf{e}_i' = \mathbf{Q}\mathbf{e}_i = Q_{mi}\mathbf{e}_m, \tag{2.16.1}$$

that is,

$$\begin{aligned} \mathbf{e}_1' &= Q_{11}\mathbf{e}_1 + Q_{21}\mathbf{e}_2 + Q_{31}\mathbf{e}_3, \\ \mathbf{e}_2' &= Q_{12}\mathbf{e}_1 + Q_{22}\mathbf{e}_2 + Q_{32}\mathbf{e}_3, \\ \mathbf{e}_3' &= Q_{13}\mathbf{e}_1 + Q_{23}\mathbf{e}_2 + Q_{33}\mathbf{e}_3, \end{aligned} \tag{2.16.2}$$

where

$$Q_{im}Q_{jm} = Q_{mi}Q_{mj} = \delta_{ij}, \tag{2.16.3}$$

or

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}. \tag{2.16.4}$$

We note that

$$\begin{aligned} Q_{11} &= \mathbf{e}_1 \cdot \mathbf{Q}\mathbf{e}_1 = \mathbf{e}_1 \cdot \mathbf{e}_1' = \text{cosine of the angle between } \mathbf{e}_1 \text{ and } \mathbf{e}_1', \\ Q_{12} &= \mathbf{e}_1 \cdot \mathbf{Q}\mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2' = \text{cosine of the angle between } \mathbf{e}_1 \text{ and } \mathbf{e}_2', \text{etc.} \end{aligned}$$

That is, in general, Q_{ij} = cosine of the angle between $\mathbf{e}_i$ and $\mathbf{e}_j'$, which may be written:

$$Q_{ij} = \cos(\mathbf{e}_i, \mathbf{e}_j'). \tag{2.16.5}$$

The matrix of these direction cosines, i.e., the matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}, \tag{2.16.6}$$

is called the *transformation matrix* between $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. Using this matrix, we shall obtain in the following sections the relationship between the two sets of components, with respect to these two sets of base vectors, of a vector and a tensor.

Example 2.16.1

Let $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ be obtained by rotating the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ about the $\mathbf{e}_3$ axis through 30°, as shown in Figure 2.16-2. In this figure, $\mathbf{e}_3$ and $\mathbf{e}'_3$ coincide.

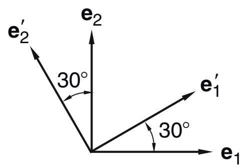

FIGURE 2.16-2

Solution

We can obtain the transformation matrix in two ways:

1. Using Eq. (2.16.5), we have

$$Q_{11} = \cos(\mathbf{e}_1, \mathbf{e}'_1) = \cos 30^\circ = \sqrt{3}/2, \quad Q_{12} = \cos(\mathbf{e}_1, \mathbf{e}'_2) = \cos 120^\circ = -1/2, \quad Q_{13} = \cos(\mathbf{e}_1, \mathbf{e}'_3) = \cos 90^\circ = 0,$$
$$Q_{21} = \cos(\mathbf{e}_2, \mathbf{e}'_1) = \cos 60^\circ = 1/2, \quad Q_{22} = \cos(\mathbf{e}_2, \mathbf{e}'_2) = \cos 30^\circ = \sqrt{3}/2, \quad Q_{23} = \cos(\mathbf{e}_2, \mathbf{e}'_3) = \cos 90^\circ = 0,$$
$$Q_{31} = \cos(\mathbf{e}_3, \mathbf{e}'_1) = \cos 90^\circ = 0, \quad Q_{32} = \cos(\mathbf{e}_3, \mathbf{e}'_2) = \cos 90^\circ = 0, \quad Q_{23} = \cos(\mathbf{e}_3, \mathbf{e}'_3) = \cos 0^\circ = 1.$$

2. It is easier to simply look at Figure 2.16-2 and decompose each of the $\mathbf{e}'_i$ into its components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ directions, i.e.,

$$\mathbf{e}'_1 = \cos 30^\circ \mathbf{e}_1 + \sin 30^\circ \mathbf{e}_2 = \frac{\sqrt{3}}{2}\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2,$$

$$\mathbf{e}'_2 = -\sin 30^\circ \mathbf{e}_1 + \cos 30^\circ \mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2,$$

$$\mathbf{e}'_3 = \mathbf{e}_3.$$

Thus, by Eq. (2.16.2), we have

$$[\mathbf{Q}] = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.17 TRANSFORMATION LAW FOR CARTESIAN COMPONENTS OF A VECTOR

Consider any vector **a**. The Cartesian components of the vector **a** with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are:

$$a_i = \mathbf{a} \cdot \mathbf{e}_i, \tag{2.17.1}$$

and its components with respect to $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are:

$$a'_i = \mathbf{a} \cdot \mathbf{e}'_i. \tag{2.17.2}$$

Now $\mathbf{e}'_i = Q_{mi}\mathbf{e}_m$ [see Eq. (2.16.1)]; therefore,

$$a'_i = \mathbf{a} \cdot Q_{mi}\mathbf{e}_m = Q_{mi}(\mathbf{a} \cdot \mathbf{e}_m), \tag{2.17.3}$$

that is,

$$a'_i = Q_{mi}a_m. \tag{2.17.4}$$

In matrix notation, Eq. (2.17.4) is

$$\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \tag{2.17.5}$$

or

$$[\mathbf{a}]' = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{a}]. \tag{2.17.6}$$

Equation (2.17.4), or Eq. (2.17.5), or Eq. (2.17.6) is the transformation law relating components of the *same* vector with respect to different rectangular Cartesian unit bases. It is very important to note that in Eq. (2.17.6), $[\mathbf{a}]'$ denotes the matrix of the vector **a** with respect to the primed basis $\{\mathbf{e}'_i\}$, and $[\mathbf{a}]$ denotes the same vector with respect to the unprimed basis $\{\mathbf{e}_i\}$. Eq. (2.17.6) is not the same as $\mathbf{a}' = \mathbf{Q}^{\mathrm{T}}\mathbf{a}$. The distinction is that $[\mathbf{a}]'$ and $[\mathbf{a}]$ are matrices of the same vector, whereas **a** and $\mathbf{a}'$ are two different vectors—$\mathbf{a}'$ being the transformed vector of **a** (through the transformation $\mathbf{a}' = \mathbf{Q}^{\mathrm{T}}\mathbf{a}$).

If we premultiply Eq. (2.17.6) with $[\mathbf{Q}]$, we get

$$[\mathbf{a}] = [\mathbf{Q}][\mathbf{a}]'. \tag{2.17.7}$$

The indicial notation for this equation is:

$$a_i = Q_{im}a'_m. \tag{2.17.8}$$

Example 2.17.1

Given that the components of a vector **a** with respect to $\{\mathbf{e}_i\}$ are given to be [2,0,0]. That is, $\mathbf{a} = 2\mathbf{e}_1$, find its components with respect to $\{\mathbf{e}'_i\}$, where the $\{\mathbf{e}'_i\}$ axes are obtained by a 90° counter-clockwise rotation of the $\{\mathbf{e}_i\}$ axis about its $\mathbf{e}_3$ axis.

Solution

The answer to the question is obvious from Figure 2.17-1, that is,

$$\mathbf{a} = 2\mathbf{e}_1 = -2\mathbf{e}'_2.$$

To show that we can get the same answer from Eq. (2.17.6), we first obtain the transformation matrix of **Q**. Since $\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = -\mathbf{e}_1$ and $\mathbf{e}'_3 = \mathbf{e}_3$, we have

$$[\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$[\mathbf{a}]' = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{a}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix},$$

that is,

$$\mathbf{a} = -2\mathbf{e}'_2.$$

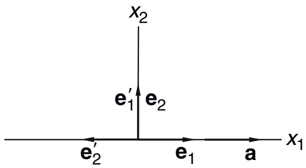

FIGURE 2.17-1

2.18 TRANSFORMATION LAW FOR CARTESIAN COMPONENTS OF A TENSOR

Consider any tensor **T**. The components of **T** with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are:

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j. \tag{2.18.1}$$

Its components with respect to $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are:

$$T'_{ij} = \mathbf{e}'_i \cdot \mathbf{T}\mathbf{e}'_j. \tag{2.18.2}$$

With $\mathbf{e}'_i = Q_{mi}\mathbf{e}_m$, we have

$$T'_{ij} = Q_{mi}\mathbf{e}_m \cdot \mathbf{T}Q_{nj}\mathbf{e}_n = Q_{mi}Q_{nj}\mathbf{e}_m \cdot \mathbf{T}\mathbf{e}_n,$$

that is,

$$T'_{ij} = Q_{mi}Q_{nj}T_{mn}. \tag{2.18.3}$$

In matrix notation, the preceding equation reads:

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}, \tag{2.18.4}$$

or

$$[\mathbf{T}]' = [\mathbf{Q}]^{\mathrm{T}} [\mathbf{T}] [\mathbf{Q}]. \tag{2.18.5}$$

We can also express the unprimed components in terms of the primed components. Indeed, if we premultiply the preceding equation with $[\mathbf{Q}]$ and post-multiply it with $[\mathbf{Q}]^{\mathrm{T}}$, we obtain, since

$$[\mathbf{Q}]\,[\mathbf{Q}]^{\mathrm{T}} = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{Q}] = [\mathbf{I}], \tag{2.18.6}$$

$$[\mathbf{T}] = [\mathbf{Q}]\,[\mathbf{T}]'\,[\mathbf{Q}]^{\mathrm{T}}. \tag{2.18.7}$$

In indicial notation, Eq. (2.18.7) reads

$$T_{ij} = Q_{im}Q_{jn}T'_{mn}. \tag{2.18.8}$$

Equations (2.18.5) [or Eq. (2.18.3)] and Eq. (2.18.7) [or Eq. (2.18.8)] are the transformation laws relating components of the *same* tensor with respect to different Cartesian unit bases. Again, it is important to note that in Eqs. (2.18.5) and (2.18.7), $[\mathbf{T}]$ and $[\mathbf{T}]'$ are different matrices of the *same* tensor $\mathbf{T}$. We note that the equation $[\mathbf{T}]' = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{T}][\mathbf{Q}]$ differs from $\mathbf{T}' = \mathbf{Q}^{\mathrm{T}}\mathbf{T}\mathbf{Q}$ in that the former relates the components of the same tensor $\mathbf{T}$ whereas the latter relates the two different tensors $\mathbf{T}$ and $\mathbf{T}'$.

Example 2.18.1

Given that with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the matrix of a tensor $\mathbf{T}$ is given by

$$[\mathrm{T}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find $[\mathbf{T}]'$, that is, find the matrix of $\mathbf{T}$ with respect to the $\mathbf{e}_i'$ basis, where $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ is obtained by rotating $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ about its $\mathbf{e}_3$-axis through 90° (see Figure 2.17-1).

Solution

Since $\mathbf{e}_1' = \mathbf{e}_2, \mathbf{e}_2' = -\mathbf{e}_1$ and $\mathbf{e}_3' = \mathbf{e}_3$, by Eq. (2.7.1) we have

$$[\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, Eq. (2.18.5) gives

$$[\mathbf{T}]' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

that is,

$$T'_{11} = 2, \quad T'_{12} = -1, \quad T'_{13} = 0, \quad T'_{22} = 0, \quad T'_{23} = 0, \quad T'_{33} = 1.$$

Example 2.18.2

Given a tensor $\mathbf{T}$ and its components T_{ij} and T'_{ij} with respect to two sets of bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$. Show that T_{ii} is invariant with respect to these bases, i.e., $T_{ii} = T'_{ii}$.

Solution

The primed components are related to the unprimed components by Eq. (2.18.3):

$$T'_{ij} = Q_{mi}Q_{nj}T_{mn},$$

thus,

$$T'_{ii} = Q_{mi} Q_{ni} T_{mn}.$$

But $Q_{mi} Q_{ni} = \delta_{mn}$ [Eq. (2.15.6)], therefore,

$$T'_{ii} = \delta_{mn} T_{mn} = T_{mm} = T_{ii},$$

that is,

$$T_{11} + T_{22} + T_{33} = T'_{11} + T'_{22} + T'_{33}.$$

We see from Example 2.18.1 that we can calculate all nine components of a tensor **T** with respect to $\{\mathbf{e}'_i\}$ from the matrix $[\mathrm{T}]_{\{\mathbf{e}_i\}}$ by using Eq. (2.18.5). However, there are often times when we need only a few components. Then it is more convenient to use Eq. (2.18.1). In matrix form, this equation is written:

$$T'_{ij} = [\mathbf{e}'_i]^{\mathrm{T}} [\mathbf{T}]\, [\mathbf{e}'_j], \tag{2.18.9}$$

where $[\mathbf{e}'_i]^{\mathrm{T}}$ denote the row matrix whose elements are the components of $\mathbf{e}'_i$ with respect to the basis $\{\mathbf{e}_i\}$.

Example 2.18.3

Obtain T'_{12} for the tensor **T** and the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ given in Example 2.18.1 by using Eq. (2.18.1).

Solution

Since $\mathbf{e}'_1 = \mathbf{e}_2$ and $\mathbf{e}'_2 = -\mathbf{e}_1$, therefore,

$$T'_{12} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_2 = \mathbf{e}_2 \cdot \mathbf{T}(-\mathbf{e}_1) = -T_{21} = -1.$$

Alternatively, using Eq. (2.18.9), we have

$$T'_{12} = [\mathbf{e}'_1]^{\mathrm{T}} [\mathbf{T}] [\mathbf{e}'_2] = [0 \quad 1 \quad 0] \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = [0 \quad 1 \quad 0] \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -1.$$

2.19 DEFINING TENSOR BY TRANSFORMATION LAWS

Equation (2.17.4) or (2.18.3) states that when the components of a vector or a tensor with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are known, then its components with respect to any $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are uniquely determined from them. In other words, the components a_i or T_{ij} with respect to one set of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ completely characterize a vector or a tensor. Thus, it is perfectly meaningful to use a statement such as "*consider a tensor* T_{ij}," meaning consider the tensor **T** whose components with respect to some set of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are T_{ij}. In fact, an alternative way of defining a tensor is through the use of transformation laws relating components of a tensor with respect to different bases. Confining ourselves to only rectangular Cartesian coordinate systems and using unit vectors along positive coordinate directions as base vectors, we now define Cartesian components of tensors of different orders in terms of their transformation laws in the following, where the primed quantities are referred to basis $\{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ and unprimed quantities to basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where the $\mathbf{e}_i'$ and $\mathbf{e}_i$ are related by $\mathbf{e}_i' = \mathbf{Q}\mathbf{e}_i$, $\mathbf{Q}$ being an orthogonal transformation:

$$\begin{array}{ll} \alpha' = \alpha & \text{zeroth-order tensor (or scalar)}, \\ a_i' = Q_{mi}a_m & \text{first-order tensor (or vector)}, \\ T_{ij}' = Q_{mi}Q_{nj}T_{mn} & \text{second-order tensor (or tensor)}, \\ S_{ijk}' = Q_{mi}Q_{nj}Q_{rk}S_{mnr} & \text{third-order tensor}, \\ C_{ijkl}' = Q_{mi}Q_{nj}Q_{rk}Q_{sl}C_{mnrs} & \text{fourth-order tensor}, \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \end{array} \tag{2.19.1}$$

Using the preceding transformation laws, we can easily establish the following three rules for tensor components: (1) the addition rule, (2) the multiplication rule, and (3) the quotient rule.

1. *The addition rule*. If T_{ij} and S_{ij} are components of any two second-order tensors, then $T_{ij} + S_{ij}$ are components of a second-order tensor. Similarly, if T_{ijk} and S_{ijk} are components of any two third-order tensors, then $T_{ijk} + S_{ijk}$ are components of a third-order tensor.

To prove this rule, we note that since $T_{ijk}' = Q_{mi}Q_{nj}Q_{rk}T_{mnr}$ and $S_{ijk}' = Q_{mi}Q_{nj}Q_{rk}S_{mnr}$, thus,

$$T_{ijk}' + S_{ijk}' = Q_{mi}Q_{nj}Q_{rk}T_{mnr} + Q_{mi}Q_{nj}Q_{rk}S_{mnr} = Q_{mi}Q_{nj}Q_{rk}(T_{mnr} + S_{mnr}).$$

Letting

$$W_{ijk}' = T_{ijk}' + S_{ijk}' \quad \text{and} \quad W_{mnr} = T_{mnr} + S_{mnr},$$

we have

$$W_{ijk}' = Q_{mi}Q_{nj}Q_{rk}W_{mnr},$$

that is, W_{ijk} are components of a third-order tensor.

2. *The multiplication rule*. Let a_i be components of any vector and T_{ij} be components of any tensor. We can form many kinds of products from these components. Examples are (a) a_ia_j, (b) $a_ia_ja_k$, (c) $T_{ij}T_{kl}$, (d) $T_{ij}T_{jk}$, etc. It can be proved that these products are components of a tensor whose order is equal to the number of free indices. For example, a_ia_j are components of a second-order tensor, $a_ia_ja_k$ are components of a third-order tensor, $T_{ij}T_{kl}$ are components of a fourth-order tensor, and $T_{ij}T_{jk}$ are components of a second-order tensor.

To prove that a_ia_j are components of a second-order tensor, we let $S_{ij} = a_ia_j$ and $S_{ij}' = a_i'a_j'$, then, since a_i are components of the vector $\mathbf{a}$, $a_i' = Q_{mi}a_m$ and $a_j' = Q_{nj}a_n$, so that

$$S_{ij}' = Q_{mi}a_mQ_{nj}a_n = Q_{mi}Q_{nj}a_ma_n = Q_{mi}Q_{nj}S_{mn},$$

thus,

$$S_{ij}' = Q_{mi}Q_{nj}S_{mn},$$

which is the transformation law for a second-order tensor.

To prove that $T_{ij}T_{kl}$ are components of a fourth-order tensor, let $M_{ijkl} = T_{ij}T_{kl}$; then we have

$$M_{ijkl}' = T_{ij}'T_{kl}' = Q_{mi}Q_{nj}T_{mn}Q_{rk}Q_{sl}T_{rs} = Q_{mi}Q_{nj}Q_{rk}Q_{sl}T_{mn}T_{rs},$$

that is,

$$M_{ijkl}' = Q_{mi}Q_{nj}Q_{rk}Q_{sl}M_{mnrs},$$

which is the transformation law for a fourth-order tensor. It is quite clear from the proofs given above that the order of the tensor whose components are obtained from the multiplication of components of tensors is determined by the number of free indices; no free index corresponds to a scalar, one free index corresponds to a vector, two free indices correspond to a second-order tensor, and so on.

3. *Quotient rule*. If a_i are components of an arbitrary vector, T_{ij} are components of an arbitrary tensor, and $a_i = T_{ij}b_j$ for all coordinates, then b_i are components of a vector.

To prove this, we note that since a_i are components of a vector and T_{ij} are components of a second-order tensor, therefore,

$$a_i = Q_{im}a'_m, \tag{i}$$

and

$$T_{ij} = Q_{im}Q_{jn}T'_{mn}. \tag{ii}$$

Now, substituting Eq. (i) and Eq. (ii) into the equation $a_i = T_{ij}b_j$, we have

$$Q_{im}a'_m = Q_{im}Q_{jn}T'_{mn}b_j. \tag{iii}$$

But the equation $a_i = T_{ij}b_j$ is true for all coordinates, thus we also have

$$a'_i = T'_{ij}b'_j \quad \text{and} \quad a'_m = T'_{mn}b'_n, \tag{iv}$$

and thus Eq. (iii) becomes

$$Q_{im}T'_{mn}b'_n = Q_{im}Q_{jn}T'_{mn}b_j. \tag{v}$$

Multiplying the preceding equation with Q_{ik} and noting that $Q_{ik}Q_{im} = \delta_{km}$, we get

$$\delta_{km}T'_{mn}b'_n = \delta_{km}Q_{jn}T'_{mn}b_j \quad \text{or} \quad T'_{kn}b'_n = Q_{jn}T'_{kn}b_j,$$

thus,

$$T'_{kn}(b'_n - Q_{jn}b_j) = 0. \tag{vi}$$

Since this equation is to be true for any tensor $\mathbf{T}$, therefore $b'_n - Q_{jn}b_j$ must be identically zero. Thus,

$$b'_n = Q_{jn}b_j. \tag{vii}$$

This is the transformation law for the components of a vector. Thus, b_i are components of a vector.

Another example that will be important later when we discuss the relationship between stress and strain for an elastic body is the following: If T_{ij} and E_{ij} are components of arbitrary second-order tensors $\mathbf{T}$ and $\mathbf{E}$, and

$$T_{ij} = C_{ijkl}E_{kl}, \tag{viii}$$

for all coordinates, then C_{ijkl} are components of a fourth-order tensor. The proof for this example follows exactly the same steps as in the previous example.

2.20 SYMMETRIC AND ANTISYMMETRIC TENSORS

A tensor is said to be symmetric if $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$. Thus, the components of a symmetric tensor have the property

$$T_{ij} = T_{ji}, \tag{2.20.1}$$

that is,

$$T_{12} = T_{21}, \quad T_{13} = T_{31}, \quad T_{23} = T_{32}. \tag{2.20.2}$$

A tensor is said to be antisymmetric if $\mathbf{T} = -\mathbf{T}^{\mathrm{T}}$. Thus the components of an antisymmetric tensor have the property

$$T_{ij} = -T_{ji}, \tag{2.20.3}$$

that is,

$$T_{11} = T_{22} = T_{33} = 0, \quad T_{12} = -T_{21}, \quad T_{13} = -T_{31}, \quad T_{23} = -T_{32}. \tag{2.20.4}$$

Any tensor $\mathbf{T}$ can always be decomposed into the sum of a symmetric tensor and an antisymmetric tensor. In fact,

$$\mathbf{T} = \mathbf{T}^{\mathrm{S}} + \mathbf{T}^{\mathrm{A}}, \tag{2.20.5}$$

where

$$\mathbf{T}^{\mathrm{S}} = \frac{\mathbf{T} + \mathbf{T}^{\mathrm{T}}}{2} \text{ is symmetric and } \mathbf{T}^{\mathrm{A}} = \frac{\mathbf{T} - \mathbf{T}^{\mathrm{T}}}{2} \text{ is anti-symmetric.} \tag{2.20.6}$$

It is not difficult to prove that the decomposition is unique (see Prob. 2.47).

2.21 THE DUAL VECTOR OF AN ANTISYMMETRIC TENSOR

The diagonal elements of an antisymmetric tensor are always zero, and, of the six nondiagonal elements, only three are independent, because $T_{12} = -T_{21}$, $T_{23} = -T_{32}$ and $T_{31} = -T_{13}$. Thus an antisymmetric tensor has really only three components, just like a vector. Indeed, it does behave like a vector. More specifically, for every antisymmetric tensor $\mathbf{T}$ there is a corresponding vector $\mathbf{t}^{\mathbf{A}}$ such that for every vector $\mathbf{a}$, the transformed vector of $\mathbf{a}$ under $\mathbf{T}$, i.e., $\mathbf{Ta}$, can be obtained from the cross-product of $\mathbf{t}^{\mathbf{A}}$ with the vector $\mathbf{a}$. That is,

$$\mathbf{Ta} = \mathbf{t}^{\mathbf{A}} \times \mathbf{a}. \tag{2.21.1}$$

This vector $\mathbf{t}^{\mathbf{A}}$ is called the *dual vector* of the antisymmetric tensor. It is also known as the *axial vector*. That such a vector indeed can be found is demonstrated here.

From Eq. (2.21.1), we have

$$\begin{aligned} T_{12} &= \mathbf{e}_1 \cdot \mathbf{Te}_2 = \mathbf{e}_1 \cdot \mathbf{t}^{\mathbf{A}} \times \mathbf{e}_2 = \mathbf{t}^{\mathbf{A}} \cdot \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{t}^{\mathbf{A}} \cdot \mathbf{e}_3 = -t_3^{\mathbf{A}}, \\ T_{31} &= \mathbf{e}_3 \cdot \mathbf{Te}_1 = \mathbf{e}_3 \cdot \mathbf{t}^{\mathbf{A}} \times \mathbf{e}_1 = \mathbf{t}^{\mathbf{A}} \cdot \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{t}^{\mathbf{A}} \cdot \mathbf{e}_2 = -t_2^{\mathbf{A}}, \\ T_{23} &= \mathbf{e}_2 \cdot \mathbf{Te}_3 = \mathbf{e}_2 \cdot \mathbf{t}^{\mathbf{A}} \times \mathbf{e}_3 = \mathbf{t}^{\mathbf{A}} \cdot \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{t}^{\mathbf{A}} \cdot \mathbf{e}_1 = -t_1^{\mathbf{A}}. \end{aligned} \tag{2.21.2}$$

Similar derivations will give $T_{21} = t_3^{\mathrm{A}}, T_{13} = t_2^{\mathrm{A}}, T_{32} = t_1^{\mathrm{A}}$ and $T_{11} = T_{22} = T_{33} = 0$. Thus, only an antisymmetric tensor has a dual vector defined by Eq. (2.21.1). It is given by

$$\mathbf{t}^{\mathbf{A}} = -(T_{23}\mathbf{e}_1 + T_{31}\mathbf{e}_2 + T_{12}\mathbf{e}_3) = T_{32}\mathbf{e}_1 + T_{13}\mathbf{e}_2 + T_{21}\mathbf{e}_3 \tag{2.21.3}$$

or, in indicial notation,

$$2\mathbf{t}^{\mathrm{A}} = -\varepsilon_{ijk}T_{jk}\mathbf{e}_i. \tag{2.21.4}$$

The calculations of dual vectors have several uses. For example, it allows us to easily obtain the axis of rotation for a finite rotation tensor. In fact, the axis of rotation is parallel to the dual vector of the antisymmetric part of the rotation tensor (see Example 2.21.2). Also, in Chapter 3 it will be shown that the dual vector can be used to obtain the infinitesimal angles of rotation of material elements under infinitesimal deformation (Section 3.11) and to obtain the angular velocity of material elements in general motion (Section 3.14).

Example 2.21.1

Given

$$[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Decompose the tensor into a symmetric and an antisymmetric part.
(b) Find the dual vector for the antisymmetric part.
(c) Verify $\mathbf{T}^A\mathbf{a} = \mathbf{t}^A \times \mathbf{a}$ for $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_3$.

Solution

(a) $[\mathbf{T}] = [\mathbf{T}^S] + [\mathbf{T}^A]$, where

$$[\mathbf{T}^S] = \frac{[\mathbf{T}] + [\mathbf{T}]^T}{2} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}, [\mathbf{T}^A] = \frac{[\mathbf{T}] - [\mathbf{T}]^T}{2} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

(b) The dual vector of $\mathbf{T}^A$ is

$$\mathbf{t}^A = -(T^A_{23}\mathbf{e}_1 + T^A_{31}\mathbf{e}_2 + T^A_{12}\mathbf{e}_3) = -(0\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_2 + \mathbf{e}_3.$$

(c) Let $\mathbf{b} = \mathbf{T}^A\mathbf{a}$. Then

$$[\mathbf{b}] = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

that is,

$$\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3.$$

We note that $\mathbf{t}^A \times \mathbf{a} = (\mathbf{e}_2 + \mathbf{e}_3) \times (\mathbf{e}_1 + \mathbf{e}_3) = -\mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_2 = \mathbf{b}$.

Example 2.21.2

Given that $\mathbf{R}$ is a rotation tensor and that $\mathbf{m}$ is a unit vector in the direction of the axis of rotation, prove that the dual vector $\mathbf{q}$ of $\mathbf{R}^A$ is parallel to $\mathbf{m}$.

Solution

Since $\mathbf{m}$ is parallel to the axis of rotation, therefore,

$$\mathbf{Rm} = \mathbf{m}.$$

Multiplying the preceding equation by $\mathbf{R}^T$ and noticing that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, we then also have the equation $\mathbf{R}^T\mathbf{m} = \mathbf{m}$. Thus,

$$(\mathbf{R} - \mathbf{R}^T)\mathbf{m} = \mathbf{0} \quad \text{or} \quad 2\mathbf{R}^A\mathbf{m} = \mathbf{0},$$

but $\mathbf{R}^A\mathbf{m} = \mathbf{q} \times \mathbf{m}$, where $\mathbf{q}$ is the dual vector of $\mathbf{R}^A$. Therefore,

$$\mathbf{q} \times \mathbf{m} = \mathbf{0}, \tag{2.21.5}$$

that is, $\mathbf{q}$ is parallel to $\mathbf{m}$. We note that it can be shown [see Prob. 2.54(b)] that if θ denotes the right-hand rotation angle, then

$$\mathbf{q} = (\sin\theta)\mathbf{m}. \tag{2.21.6}$$

2.22 EIGENVALUES AND EIGENVECTORS OF A TENSOR

Consider a tensor $\mathbf{T}$. If $\mathbf{a}$ is a vector that transforms under $\mathbf{T}$ into a vector parallel to itself, that is,

$$\mathbf{Ta} = \lambda\mathbf{a}, \tag{2.22.1}$$

then $\mathbf{a}$ is an *eigenvector* and λ is the corresponding *eigenvalue*.

If $\mathbf{a}$ is an eigenvector with corresponding eigenvalue λ of the linear transformation $\mathbf{T}$, any vector parallel to $\mathbf{a}$ is also an eigenvector with the same eigenvalue λ. In fact, for any scalar α

$$\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{Ta} = \alpha(\lambda\mathbf{a}) = \lambda(\alpha\mathbf{a}). \tag{2.22.2}$$

Thus, an eigenvector, as defined by Eq. (2.22.1), has an arbitrary length. For definiteness, *we shall agree that all eigenvectors sought will be of unit length*.

A tensor may have infinitely many eigenvectors. In fact, since $\mathbf{Ia} = \mathbf{a}$, any vector is an eigenvector for the identity tensor $\mathbf{I}$, with eigenvalues all equal to unity. For the tensor $\beta\mathbf{I}$, the same is true except that the eigenvalues are all equal to β.

Some tensors only have eigenvectors in one direction. For example, for any rotation tensor that effects a rigid body rotation about an axis through an angle not equal to an integral multiple of π, only those vectors that are parallel to the axis of rotation will remain parallel to themselves.

Let $\mathbf{n}$ be a unit eigenvector. Then

$$\mathbf{Tn} = \lambda\mathbf{n} = \lambda\mathbf{In}, \tag{2.22.3}$$

thus,

$$(\mathbf{T} - \lambda\mathbf{I})\mathbf{n} = \mathbf{0} \quad \text{with} \quad \mathbf{n} \cdot \mathbf{n} = 1. \tag{2.22.4}$$

Let $\mathbf{n} = \alpha_i\mathbf{e}_i$; then, in component form,

$$\left(T_{ij} - \lambda\delta_{ij}\right)\alpha_j = 0 \quad \text{with} \quad \alpha_j\alpha_j = 1. \tag{2.22.5}$$

In long form, we have

$$\begin{aligned}(T_{11} - \lambda)\alpha_1 + T_{12}\alpha_2 + T_{13}\alpha_3 &= 0,\\ T_{21}\alpha_1 + (T_{22} - \lambda)\alpha_2 + T_{23}\alpha_3 &= 0,\\ T_{31}\alpha_1 + T_{32}\alpha_2 + (T_{33} - \lambda)\alpha_3 &= 0.\end{aligned} \tag{2.22.6}$$

Equations (2.22.6) are a system of linear homogeneous equations in α_1, α_2 and α_3. Obviously, a solution for this system is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. This is known as the *trivial solution*. This solution simply states the obvious fact that $\mathbf{a} = \mathbf{0}$ satisfies the equation $\mathbf{Ta} = \lambda\mathbf{a}$, independent of the value of λ. To find the nontrivial eigenvectors for $\mathbf{T}$, we note that a system of homogeneous, linear equations admits a nontrivial solution only if the determinant of its coefficients vanishes. That is,

$$|\mathbf{T} - \lambda\mathbf{I}| = 0, \tag{2.22.7}$$

that is,

$$\begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = 0. \tag{2.22.8}$$

Expanding the determinant results in a cubic equation in λ. It is called the *characteristic equation* of $\mathbf{T}$. The roots of this characteristic equation are the *eigenvalues* of $\mathbf{T}$.

Equations (2.22.6), together with the equation

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \tag{2.22.9}$$

allow us to obtain eigenvectors of unit length. The procedure for finding the eigenvalues and eigenvectors of a tensor are best illustrated by example.

Example 2.22.1

Find the eigenvalues and eigenvectors for the tensor whose components are

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution

We note that this tensor is $2\mathbf{I}$, so that $\mathbf{Ta} = 2\mathbf{Ia} = 2\mathbf{a}$ for any vector $\mathbf{a}$. Therefore, by the definition of eigenvector [see Eq. (2.22.1)], any direction is a direction for an eigenvector. The eigenvalue for every direction is the same, which is 2. However, we can also use Eq. (2.22.8) to find the eigenvalues and Eqs. (2.22.6) to find the eigenvectors. Indeed, Eq. (2.22.8) gives, for this tensor, the following characteristic equation:

$$(2 - \lambda)^3 = 0,$$

so we have a triple root $\lambda = 2$. Substituting this value in Eqs. (2.22.6), we have

$$(2 - 2)\alpha_1 = 0, \quad (2 - 2)\alpha_2 = 0, \quad (2 - 2)\alpha_3 = 0.$$

Thus, all three equations are automatically satisfied for arbitrary values of α_1, α_2 and α_3 so that every direction is a direction for an eigenvector. We can choose any three noncoplanar directions as the three independent eigenvectors; on them all other eigenvectors depend. In particular, we can choose $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as a set of independent eigenvectors.

Example 2.22.2

Show that if $T_{21} = T_{31} = 0$, then $\pm\mathbf{e}_1$ are eigenvectors of $\mathbf{T}$ with eigenvalue T_{11}.

Solution

From $\mathbf{Te}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3$, we have

$$\mathbf{Te}_1 = T_{11}\mathbf{e}_1 \quad \text{and} \quad \mathbf{T}(-\mathbf{e}_1) = T_{11}(-\mathbf{e}_1).$$

Thus, by definition, Eq. (2.22.1), $\pm\mathbf{e}_1$ are eigenvectors with T_{11} as its eigenvalue. Similarly, if $T_{12} = T_{32} = 0$, then $\pm\mathbf{e}_2$ are eigenvectors with corresponding eigenvalue T_{22}, and if $T_{13} = T_{23} = 0$, then $\pm\mathbf{e}_3$ are eigenvectors with corresponding eigenvalue T_{33}.

Example 2.22.3

Given that

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Find the eigenvalues and their corresponding eigenvectors.

Solution

The characteristic equation is

$$(2-\lambda)^2(3-\lambda) = 0.$$

Thus, $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 2$ (obviously the ordering of the eigenvalues is arbitrary). These results are obvious in view of Example 2.22.2. In fact, that example also tells us that the eigenvectors corresponding to $\lambda_1 = 3$ are $\pm\mathbf{e}_3$ and eigenvectors corresponding to $\lambda_2 = \lambda_3 = 2$ are $\pm\mathbf{e}_1$ and $\pm\mathbf{e}_2$. However, there are actually infinitely many eigenvectors corresponding to the double root. In fact, since

$$\mathbf{Te}_1 = 2\mathbf{e}_1 \quad \text{and} \quad \mathbf{Te}_2 = 2\mathbf{e}_2,$$

therefore, for any α and β,

$$\mathbf{T}(\alpha\mathbf{e}_1 + \beta\mathbf{e}_2) = \alpha\mathbf{Te}_1 + \beta\mathbf{Te}_2 = 2\alpha\mathbf{e}_1 + 2\beta\mathbf{e}_2 = 2(\alpha\mathbf{e}_1 + \beta\mathbf{e}_2),$$

that is, $\alpha\mathbf{e}_1 + \beta\mathbf{e}_2$ is an eigenvector with eigenvalue 2. This fact can also be obtained from Eqs. (2.22.6). With $\lambda = 2$, these equations give

$$0\alpha_1 = 0, \quad 0\alpha_2 = 0, \quad \alpha_3 = 0.$$

Thus, α_1 = arbitrary, α_2 = arbitrary, and $\alpha_3 = 0$, so that any vector perpendicular to $\mathbf{e}_3$, that is, any $\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$, is an eigenvector.

Example 2.22.4

Find the eigenvalues and eigenvectors for the tensor

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}.$$

Solution

The characteristic equation gives

$$|\mathbf{T}-\lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & -3-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2-25) = 0.$$

Thus, there are three distinct eigenvalues, $\lambda_1 = 2$, $\lambda_2 = 5$ and $\lambda_3 = -5$.

Corresponding to $\lambda_1 = 2$, Eqs. (2.22.6) gives

$$0\alpha_1 = 0, \quad \alpha_2 + 4\alpha_3 = 0, \quad 4\alpha_2 - 5\alpha_3 = 0,$$

and we also have Eq. (2.22.9):

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1.$$

Thus, $\alpha_2 = \alpha_3 = 0$ and $\alpha_1 = \pm 1$ so that the eigenvector corresponding to $\lambda_1 = 2$ is

$$\mathbf{n}_1 = \pm\mathbf{e}_1.$$

We note that from the Example 2.22.2, this eigenvalue 2 and the corresponding eigenvectors $\mathbf{n}_1 = \pm\mathbf{e}_1$ can be written by inspection.

Corresponding to $\lambda_2 = 5$, we have

$$-3\alpha_1 = 0, \quad -2\alpha_2 + 4\alpha_3 = 0, \quad 4\alpha_2 - 8\alpha_3 = 0,$$

thus (note the second and third equations are the same),

$$\alpha_1 = 0, \ \alpha_2 = 2\alpha_3,$$

and the unit eigenvectors corresponding to $\lambda_2 = 5$ are

$$\mathbf{n}_2 = \pm\frac{1}{\sqrt{5}}(2\mathbf{e}_2 + \mathbf{e}_3).$$

Similarly for $\lambda_3 = -5$, the unit eigenvectors are

$$\mathbf{n}_3 = \pm\frac{1}{\sqrt{5}}(-\mathbf{e}_2 + 2\mathbf{e}_3).$$

All the examples given here have three eigenvalues that are real. It can be shown that if a tensor is real (i.e., with real components) and symmetric, then all its eigenvalues are real. If a tensor is real but not symmetric, then two of the eigenvalues may be complex conjugates. The following is such an example.

Example 2.22.5

Find the eigenvalues and eigenvectors for the rotation tensor **R** corresponding to a 90° rotation about the $\mathbf{e}_3$ (see Example 2.10.1).

Solution

With

$$[\mathbf{R}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the characteristic equation is

$$\begin{vmatrix} 0-\lambda & -1 & 0 \\ 1 & 0-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0,$$

that is,

$$\lambda^2(1-\lambda) + (1-\lambda) = (1-\lambda)(\lambda^2+1) = 0.$$

Thus, only one eigenvalue is real, namely $\lambda_1 = 1$; the other two, $\lambda_2 = +\sqrt{-1}$ and $\lambda_3 = -\sqrt{-1}$, are imaginary. Only real eigenvalues are of interest to us. We shall therefore compute only the eigenvector corresponding to $\lambda_1 = 1$. From

$$(0-1)\alpha_1 - \alpha_2 = 0, \quad \alpha_1 - \alpha_2 = 0, \quad (1-1)\alpha_3 = 0,$$

and

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$

we obtain $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = \pm 1$, that is,

$$\mathbf{n} = \pm\mathbf{e}_3,$$

which, of course, are parallel to the axis of rotation.

2.23 PRINCIPAL VALUES AND PRINCIPAL DIRECTIONS OF REAL SYMMETRIC TENSORS

In the following chapters, we shall encounter several real tensors (stress tensor, strain tensor, rate of deformation tensor, etc.) that are symmetric. The following significant theorem can be proven: *The eigenvalues of any real symmetric tensor are all real* (we omit the proof). Thus, for a real symmetric tensor, there always exist at least three real eigenvectors, which we shall also call the *principal directions*. The corresponding eigenvalues are called the *principal values*.

We now prove that there always exist three principal directions that are mutually perpendicular. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be two eigenvectors corresponding to the eigenvalues λ_1 and λ_2, respectively, of a tensor **T**. Then

$$\mathbf{T}\mathbf{n}_1 = \lambda_1\mathbf{n}_1, \tag{2.23.1}$$

and

$$\mathbf{T}\mathbf{n}_2 = \lambda_2\mathbf{n}_2. \tag{2.23.2}$$

Thus,

$$\mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_1 = \lambda_1\mathbf{n}_2 \cdot \mathbf{n}_1, \tag{2.23.3}$$

and

$$\mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 = \lambda_2 \mathbf{n}_1 \cdot \mathbf{n}_2. \tag{2.23.4}$$

For a symmetric tensor, $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$, so that

$$\mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 = \mathbf{n}_2 \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n}_1 = \mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_1. \tag{2.23.5}$$

Thus, from Eqs. (2.23.3) and (2.23.4), we have

$$(\lambda_1 - \lambda_2)(\mathbf{n}_1 \cdot \mathbf{n}_2) = 0. \tag{2.23.6}$$

It follows that if λ_1 is not equal to λ_2, then $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$, that is, $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular to each other. We have thus proved that if the eigenvalues of a symmetric tensor are all distinct, then *the three principal directions are mutually perpendicular*.

Next, let us suppose that $\mathbf{n}_1$ and $\mathbf{n}_2$ are two eigenvectors corresponding to the same eigenvalue λ. Then, by definition, $\mathbf{T}\mathbf{n}_1 = \lambda\mathbf{n}_1$ and $\mathbf{T}\mathbf{n}_2 = \lambda\mathbf{n}_2$ so that for any α and β,

$$\mathbf{T}(\alpha\mathbf{n}_1 + \beta\mathbf{n}_2) = \alpha\mathbf{T}\mathbf{n}_1 + \beta\mathbf{T}\mathbf{n}_2 = \alpha\lambda\mathbf{n}_1 + \beta\lambda\mathbf{n}_2 = \lambda(\alpha\mathbf{n}_1 + \beta\mathbf{n}_2).$$

That is, $(\alpha\mathbf{n}_1 + \beta\mathbf{n}_2)$ is also an eigenvector with the same eigenvalue λ. In other words, if there are two distinct eigenvectors with the same eigenvalue, then there are infinitely many eigenvectors (which form a plane) with the same eigenvalue. This situation arises when the characteristic equation has a repeated root (see Example 2.22.3). Suppose the characteristic equation has roots $\lambda_1 = \lambda_2 = \lambda$ and λ_3 (λ_3 distinct from λ). Let $\mathbf{n}_3$ be the eigenvector corresponding to λ_3; then $\mathbf{n}_3$ is perpendicular to any eigenvector of λ. Therefore there exist infinitely many sets of three mutually perpendicular principal directions, each containing $\mathbf{n}_3$ and any two mutually perpendicular eigenvectors of the repeated root λ.

In the case of a triple root, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, any vector is an eigenvector (see Example 2.22.1) so that there exist infinitely many sets of three mutually perpendicular principal directions.

From these discussions, we conclude that for every real symmetric tensor there exists at least one triad of principal directions that are mutually perpendicular.

2.24 MATRIX OF A TENSOR WITH RESPECT TO PRINCIPAL DIRECTIONS

We have shown that for a real symmetric tensor, there always exist three principal directions that are mutually perpendicular. Let $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ be unit vectors in these directions. Then, using $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ as base vectors, the components of the tensor are

$$\begin{aligned}
T_{11} &= \mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_1 = \mathbf{n}_1 \cdot \lambda_1\mathbf{n}_1 = \lambda_1\mathbf{n}_1 \cdot \mathbf{n}_1 = \lambda_1,\\
T_{22} &= \mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_2 = \mathbf{n}_2 \cdot \lambda_2\mathbf{n}_2 = \lambda_2\mathbf{n}_2 \cdot \mathbf{n}_2 = \lambda_2,\\
T_{33} &= \mathbf{n}_3 \cdot \mathbf{T}\mathbf{n}_3 = \mathbf{n}_3 \cdot \lambda_3\mathbf{n}_3 = \lambda_3\mathbf{n}_3 \cdot \mathbf{n}_3 = \lambda_3,\\
T_{12} &= \mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 = \mathbf{n}_1 \cdot \lambda_2\mathbf{n}_2 = \lambda_2\mathbf{n}_1 \cdot \mathbf{n}_2 = 0,\\
T_{13} &= \mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_3 = \mathbf{n}_1 \cdot \lambda_3\mathbf{n}_3 = \lambda_3\mathbf{n}_1 \cdot \mathbf{n}_3 = 0,\\
T_{23} &= \mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_3 = \mathbf{n}_2 \cdot \lambda_3\mathbf{n}_3 = \lambda_3\mathbf{n}_2 \cdot \mathbf{n}_3 = 0,
\end{aligned} \tag{2.24.1}$$

that is,

$$[\mathbf{T}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_{\mathbf{n}_i}. \tag{2.24.2}$$

Thus, the matrix is diagonal and the diagonal elements are the eigenvalues of $\mathbf{T}$.

We now show that the principal values of a tensor $\mathbf{T}$ include the maximum and the minimum values that the diagonal elements of any matrix of $\mathbf{T}$ can have. First, for any unit vector $\mathbf{e}_1' = \alpha\mathbf{n}_1 + \beta\mathbf{n}_2 + \gamma\mathbf{n}_3$,

$$T'_{11} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_1 = [\alpha \quad \beta \quad \gamma] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \tag{2.24.3}$$

that is,

$$T'_{11} = \lambda_1 \alpha^2 + \lambda_2 \beta^2 + \lambda_3 \gamma^2. \tag{2.24.4}$$

Without loss of generality, let

$$\lambda_1 \geq \lambda_2 \geq \lambda_3. \tag{2.24.5}$$

Then, noting that $\alpha^2 + \beta^2 + \gamma^2 = 1$, we have

$$\lambda_1 = \lambda_1(\alpha^2 + \beta^2 + \gamma^2) \geq \lambda_1 \alpha^2 + \lambda_2 \beta^2 + \lambda_3 \gamma^2, \tag{2.24.6}$$

that is,

$$\lambda_1 \geq T'_{11}. \tag{2.24.7}$$

We also have

$$\lambda_1 \alpha^2 + \lambda_2 \beta^2 + \lambda_3 \gamma^2 \geq \lambda_3(\alpha^2 + \beta^2 + \gamma^2) = \lambda_3, \tag{2.24.8}$$

that is,

$$T'_{11} \geq \lambda_3. \tag{2.24.9}$$

Thus, the maximum value of the principal values of **T** is the maximum value of the diagonal elements of *all matrices* of **T**, and the minimum value of the principal values of **T** is the minimum value of the diagonal elements of *all matrices* of **T**. It is important to remember that for a given **T**, there are infinitely many matrices and therefore, infinitely many diagonal elements, of which the maximum principal value is the maximum of all of them and the minimum principal value is the minimum of all of them.

2.25 PRINCIPAL SCALAR INVARIANTS OF A TENSOR

The characteristic equation of a tensor **T**, $|T_{ij} - \lambda \delta_{ij}| = 0$ can be written as:

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0, \tag{2.25.1}$$

where

$$I_1 = T_{11} + T_{22} + T_{33} = T_{ii} = \text{tr}\mathbf{T}, \tag{2.25.2}$$

$$I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} = \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ji}\right) = \frac{1}{2}\left[(tr\mathbf{T})^2 - \text{tr}(\mathbf{T}^2)\right], \tag{2.25.3}$$

$$I_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = \det[\mathbf{T}]. \tag{2.25.4}$$

Since by definition, the eigenvalues of **T** do not depend on the choices of the base vectors, therefore the coefficients of Eq. (2.25.1) will not depend on any particular choices of basis. They are called the *principal scalar invariants* of **T**.

We note that, in terms of the eigenvalues of **T**, which are the roots of Eq. (2.25.1), the scalar invariants take the simple form

$$\begin{aligned} I_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \\ I_3 &= \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{2.25.5}$$

Example 2.25.1

For the tensor of Example 2.22.4, first find the principal scalar invariants and then evaluate the eigenvalues using Eq. (2.25.1).

Solution

The matrix of **T** is

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}.$$

Thus,

$$\begin{aligned} I_1 &= 2 + 3 - 3 = 2, \\ I_2 &= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -25, \\ I_3 &= |\mathbf{T}| = -50. \end{aligned}$$

These values give the characteristic equation as

$$\lambda^3 - 2\lambda^2 - 25\lambda + 50 = 0,$$

or

$$(\lambda - 2)(\lambda - 5)(\lambda + 5) = 0.$$

Thus the eigenvalues are $\lambda = 2$, $\lambda = 5$ and $\lambda = -5$, as previously determined.

PROBLEMS FOR PART B

2.19 A transformation **T** operates on any vector **a** to give $\mathbf{Ta} = \mathbf{a}/|\mathbf{a}|$, where $|\mathbf{a}|$ is the magnitude of **a**. Show that **T** is not a linear transformation.

2.20 (a) A tensor **T** transforms every vector **a** into a vector $\mathbf{Ta} = \mathbf{m} \times \mathbf{a}$, where **m** is a specified vector. Show that **T** is a linear transformation. (b) If $\mathbf{m} = \mathbf{e}_1 + \mathbf{e}_2$, find the matrix of the tensor **T**.

2.21 A tensor **T** transforms the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ such that $\mathbf{Te}_1 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{Te}_2 = \mathbf{e}_1 - \mathbf{e}_2$. If $\mathbf{a} = 2\mathbf{e}_1 + 3\mathbf{e}_2$ and $\mathbf{b} = 3\mathbf{e}_1 + 2\mathbf{e}_2$, use the linear property of **T** to find (a) **Ta**, (b) **Tb**, and (c) **T(a+b)**.

2.22 Obtain the matrix for the tensor **T**, that transforms the base vectors as follows: $\mathbf{Te}_1 = 2\mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{Te}_2 = \mathbf{e}_2 + 3\mathbf{e}_3$, $\mathbf{Te}_3 = -\mathbf{e}_1 + 3\mathbf{e}_2$.

2.23 Find the matrix of the tensor **T** that transforms any vector **a** into a vector $\mathbf{b} = \mathbf{m}(\mathbf{a} \cdot \mathbf{n})$ where $\mathbf{m} = \frac{\sqrt{2}}{2}(\mathbf{e}_1 + \mathbf{e}_2)$ and $\mathbf{n} = \frac{\sqrt{2}}{2}(-\mathbf{e}_1 + \mathbf{e}_3)$.

2.24 (a) A tensor **T** transforms every vector into its mirror image with respect to the plane whose normal is $\mathbf{e}_2$. Find the matrix of **T**. (b) Do part (a) if the plane has a normal in the $\mathbf{e}_3$ direction.

2.25 (a) Let **R** correspond to a right-hand rotation of angle θ about the x_1-axis. Find the matrix of **R**. (b) Do part (a) if the rotation is about the x_2-axis. The coordinates are right-handed.

2.26 Consider a plane of reflection that passes through the origin. Let **n** be a unit normal vector to the plane and let **r** be the position vector for a point in space. (a) Show that the reflected vector for **r** is given by $\mathbf{Tr} = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n}$, where **T** is the transformation that corresponds to the reflection. (b) Let $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$; find the matrix of **T**. (c) Use this linear transformation to find the mirror image of the vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$.

2.27 Knowing that the reflected vector for **r** is given by $\mathbf{Tr} = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n}$ (see the previous problem), where **T** is the transformation that corresponds to the reflection and **n** is the normal to the mirror, show that in dyadic notation the reflection tensor is given by $\mathbf{T} = \mathbf{I} - 2\mathbf{nn}$ and find the matrix of **T** if the normal of the mirror is given by $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$.

2.28 A rotation tensor **R** is defined by the relation $\mathbf{Re}_1 = \mathbf{e}_2,\ \mathbf{Re}_2 = \mathbf{e}_3,\ \mathbf{Re}_3 = \mathbf{e}_1$. (a) Find the matrix of **R** and verify that $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$ and (b) find a unit vector in the direction of the axis of rotation that could have been used to effect this particular rotation.

2.29 A rigid body undergoes a right-hand rotation of angle θ about an axis that is in the direction of the unit vector **m**. Let the origin of the coordinates be on the axis of rotation and **r** be the position vector for a typical point in the body. (a) Show that the rotated vector of **r** is given by: $\mathbf{Rr} = (1 - \cos\theta)(\mathbf{m} \cdot \mathbf{r})\mathbf{m} + \cos\theta\,\mathbf{r} + \sin\theta\,(\mathbf{m} \times \mathbf{r})$, where **R** is the rotation tensor. (b) Let $\mathbf{m} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$, find the matrix for **R**.

2.30 For the rotation about an arbitrary axis **m** by an angle θ, (a) show that the rotation tensor is given by $\mathbf{R} = (1 - \cos\theta)(\mathbf{mm}) + \cos\theta\,\mathbf{I} + \mathbf{sin}\,\theta\,\mathbf{E}$, where **mm** denotes that dyadic product of **m** and **m**, and **E** is the antisymmetric tensor whose dual vector (or axial vector) is **m**, (b) find $\mathbf{R}^{\mathrm{A}}$, the antisymmetric part of **R** and (c) show that the dual vector for $\mathbf{R}^{\mathrm{A}}$ is given by $(\sin\,\theta)\mathbf{m}$. *Hint:* $\mathbf{Rr} = (1 - \cos\theta)(\mathbf{m} \cdot \mathbf{r})\mathbf{m} + \cos\theta\,\mathbf{r} + \sin\theta\,(\mathbf{m} \times \mathbf{r})$ (see previous problem).

2.31 (a) Given a mirror whose normal is in the direction of $\mathbf{e}_2$, find the matrix of the tensor **S**, which first transforms every vector into its mirror image and then transforms them by a 45° right-hand rotation about the $\mathbf{e}_1$-axis. (b) Find the matrix of the tensor **T**, which first transforms every vector by a 45° right-hand rotation about the $\mathbf{e}_1$-axis and then transforms them by a reflection with respect to a mirror (with normal $\mathbf{e}_2$). (c) Consider the vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$; find the transformed vector by using the transformation **S**. (d) For the same vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$, find the transformed vector by using the transformation **T**.

2.32 Let **R** correspond to a right-hand rotation of angle θ about the x_3-axis; (a) find the matrix of $\mathbf{R}^2$. (b) Show that $\mathbf{R}^2$ corresponds to a rotation of angle 2θ about the same axis. (c) Find the matrix of $\mathbf{R}^n$ for any integer n.

2.33 Rigid body rotations that are small can be described by an orthogonal transformation $\mathbf{R} = \mathbf{I} + \varepsilon\mathbf{R}^*$, where $\varepsilon \to 0$ as the rotation angle approaches zero. Consider two successive small rotations, $\mathbf{R}_1$ and $\mathbf{R}_2$; show that the final result does not depend on the order of rotations.

2.34 Let **T** and **S** be any two tensors. Show that (a) $\mathbf{T}^{\mathrm{T}}$ is a tensor, (b) $\mathbf{T}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}} = (\mathbf{T} + \mathbf{S})^{\mathrm{T}}$, and (c) $(\mathbf{TS})^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$.

2.35 For arbitrary tensors **T** and **S**, without relying on the component form, prove that (a) $(\mathbf{T}^{-1})^{\mathrm{T}} = (\mathbf{T}^{\mathrm{T}})^{-1}$ and (b) $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$.

2.36 Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$ be two rectangular Cartesian base vectors. (a) Show that if $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$, then $\mathbf{e}_i = Q_{im}\mathbf{e}_m'$. (b) Verify $Q_{mi}Q_{mj} = \delta_{ij} = Q_{im}Q_{jm}$.

2.37 The basis $\{\mathbf{e}_i'\}$ is obtained by a 30° counterclockwise rotation of the $\{\mathbf{e}_i\}$ basis about the $\mathbf{e}_3$ axis. (a) Find the transformation matrix $[\mathbf{Q}]$ relating the two sets of basis. (b) By using the vector transformation law, find the components of $\mathbf{a} = \sqrt{3}\mathbf{e}_1 + \mathbf{e}_2$ in the primed basis, i.e., find a_i' and (c) do part (b) geometrically.

2.38 Do the previous problem with the $\{\mathbf{e}_i'\}$ basis obtained by a 30° clockwise rotation of the $\{\mathbf{e}_i\}$ basis about the $\mathbf{e}_3$ axis.

2.39 The matrix of a tensor $\mathbf{T}$ with respect to the basis $\{\mathbf{e}_i\}$ is

$$[\mathbf{T}] = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}.$$

Find T_{11}', T_{12}' and T_{31}' with respect to a right-handed basis $\{\mathbf{e}_i'\}$ where $\mathbf{e}_1'$ is in the direction of $-\mathbf{e}_2 + 2\mathbf{e}_3$ and $\mathbf{e}_2'$ is in the direction of $\mathbf{e}_1$.

2.40 (a) For the tensor of the previous problem, find $[T_{ij}']$, i.e., $[\mathbf{T}]_{\mathbf{e}_i'}$ where $\{\mathbf{e}_i'\}$ is obtained by a 90° right-hand rotation about the $\mathbf{e}_3$ axis and (b) obtain T_{ii}' and the determinant $|T_{ij}'|$ and compare them with T_{ii} and $|T_{ij}|$.

2.41 The dot product of two vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$ is equal to a_ib_i. Show that the dot product is a scalar invariant with respect to orthogonal transformations of coordinates.

2.42 If T_{ij} are the components of a tensor, (a) show that $T_{ij}T_{ij}$ is a scalar invariant with respect to orthogonal transformations of coordinates, (b) evaluate $T_{ij}T_{ij}$ with respect to the basis $\{\mathbf{e}_i\}$ for $[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{\mathbf{e}_i}$, (c) find $[\mathbf{T}]'$ if $\mathbf{e}_i' = \mathbf{Q}\mathbf{e}_i$, where $[\mathbf{Q}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{\mathbf{e}_i}$, and (d) verify for the above that $T_{ij}'T_{ij}' = T_{ij}T_{ij}$.

2.43 Let $[\mathbf{T}]$ and $[\mathbf{T}]'$ be two matrices of the same tensor $\mathbf{T}$. Show that $\det[\mathbf{T}] = \det[\mathbf{T}]'$.

2.44 (a) If the components of a third-order tensor are R_{ijk}, show that R_{iik} are components of a vector. (b) If the components of a fourth-order tensor are R_{ijkl}, show that R_{iikl} are components of a second-order tensor. (c) What are components of $R_{iik\ldots}$ if $R_{ijk\ldots}$ are components of a tensor of n^{th} order?

2.45 The components of an arbitrary vector $\mathbf{a}$ and an arbitrary second tensor $\mathbf{T}$ are related by a triply subscripted quantity R_{ijk} in the manner $a_i = R_{ijk}T_{jk}$ for any rectangular Cartesian basis $\{\mathbf{e}_i\}$. Prove that R_{ijk} are the components of a third-order tensor.

2.46 For any vector $\mathbf{a}$ and any tensor $\mathbf{T}$, show that (a) $\mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} = 0$ and (b) $\mathbf{a}\cdot\mathbf{T}\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^{\mathrm{S}}\mathbf{a}$, where $\mathbf{T}^{\mathrm{A}}$ and $\mathbf{T}^{\mathrm{S}}$ are antisymmetric and symmetric part of $\mathbf{T}$, respectively.

2.47 Any tensor can be decomposed into a symmetric part and an antisymmetric part, that is, $\mathbf{T} = \mathbf{T}^{\mathrm{S}} + \mathbf{T}^{\mathrm{A}}$. Prove that the decomposition is unique. (*Hint:* Assume that it is not true and show contradiction.)

2.48 Given that a tensor $\mathbf{T}$ has the matrix $[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, (a) find the symmetric part and the antisymmetric part of $\mathbf{T}$ and (b) find the dual vector (or axial vector) of the antisymmetric part of $\mathbf{T}$.

2.49 Prove that the only possible real eigenvalues of an orthogonal tensor $\mathbf{Q}$ are $\lambda = \pm 1$. Explain the direction of the eigenvectors corresponding to them for a proper orthogonal (rotation) tensor and for an improper orthogonal (reflection) tensor.

2.50 Given the improper orthogonal tensor $[\mathbf{Q}] = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$. (a) Verify that det $[\mathbf{Q}] = -1$. (b) Verify that the eigenvalues are $\lambda = 1$ and -1. (c) Find the normal to the plane of reflection (i.e., eigenvectors corresponding to $\lambda = -1$) and (d) find the eigenvectors corresponding to $\lambda = 1$ (vectors parallel to the plane of reflection).

2.51 Given that tensors $\mathbf{R}$ and $\mathbf{S}$ have the same eigenvector $\mathbf{n}$ and corresponding eigenvalues r_1 and s_1, respectively, find an eigenvalue and the corresponding eigenvector for $\mathbf{T} = \mathbf{RS}$.

2.52 Show that if $\mathbf{n}$ is a real eigenvector of an antisymmetric tensor $\mathbf{T}$, then the corresponding eigenvalue vanishes.

2.53 (a) Show that $\mathbf{a}$ is an eigenvector for the dyadic product $\mathbf{ab}$ of vectors $\mathbf{a}$ and $\mathbf{b}$ with eigenvalue $\mathbf{a} \cdot \mathbf{b}$, (b) find the first principal scalar invariant of the dyadic product $\mathbf{ab}$ and (c) show that the second and the third principal scalar invariant of the dyadic product $\mathbf{ab}$ vanish, and that zero is a double eigenvalue of $\mathbf{ab}$.

2.54 For any rotation tensor, a set of basis $\{\mathbf{e}'_i\}$ may be chosen with $\mathbf{e}'_3$ along the axis of rotation so that $\mathbf{Re}'_1 = \cos\theta\,\mathbf{e}'_1 + \sin\theta\,\mathbf{e}'_2$, $\mathbf{Re}'_2 = -\sin\theta\,\mathbf{e}'_1 + \cos\theta\,\mathbf{e}'_2$, $\mathbf{Re}'_3 = \mathbf{e}'_3$, where θ is the angle of right-hand rotation. (a) Find the antisymmetric part of $\mathbf{R}$ with respect to the basis $\{\mathbf{e}'_i\}$, i.e., find $[\mathbf{R}^A]_{\mathbf{e}'_i}$. (b) Show that the dual vector of $\mathbf{R}^A$ is given by $\mathbf{t}^A = \sin\theta\,\mathbf{e}'_3$ and (c) show that the first scalar invariant of $\mathbf{R}$ is given by $1 + 2\cos\theta$. That is, for any given rotation tensor $\mathbf{R}$, its axis of rotation and the angle of rotation can be obtained from the dual vector of $\mathbf{R}^A$ and the first scalar invariant of $\mathbf{R}$.

2.55 The rotation of a rigid body is described by $\mathbf{Re}_1 = \mathbf{e}_2$, $\mathbf{Re}_2 = \mathbf{e}_3$, $\mathbf{Re}_3 = \mathbf{e}_1$. Find the axis of rotation and the angle of rotation. Use the result of the previous problem.

2.56 Given the tensor $[\mathbf{Q}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (a) Show that the given tensor is a rotation tensor. (b) Verify that the eigenvalues are $\lambda = 1$ and -1. (c) Find the direction for the axis of rotation (i.e., eigenvectors corresponding to $\lambda = 1$). (d) Find the eigenvectors corresponding to $\lambda = -1$ and (e) obtain the angle of rotation using the formula $I_1 = 1 + 2\cos\theta$ (see Prob. 2.54), where I_1 is the first scalar invariant of the rotation tensor.

2.57 Let $\mathbf{F}$ be an arbitrary tensor. (a) Show that $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$ are both symmetric tensors. (b) If $\mathbf{F} = \mathbf{QU} = \mathbf{VQ}$, where $\mathbf{Q}$ is orthogonal, show that $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ and $\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$. (c) If λ and $\mathbf{n}$ are eigenvalue and the corresponding eigenvector for $\mathbf{U}$, find the eigenvalue and eigenvector for $\mathbf{V}$.

2.58 Verify that the second principal scalar invariant of a tensor $\mathbf{T}$ can be written: $I_2 = \dfrac{T_{ii}T_{jj}}{2} - \dfrac{T_{ij}T_{ji}}{2}$.

2.59 A tensor $\mathbf{T}$ has a matrix $[\mathbf{T}]$ given below. (a) Write the characteristic equation and find the principal values and their corresponding principal directions. (b) Find the principal scalar invariants. (c) If $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are the principal directions, write $[\mathbf{T}]_{\mathbf{n}_i}$. (d) Could the following matrix $[\mathbf{S}]$ represent the same tensor $\mathbf{T}$ with respect to some basis? $[\mathbf{T}] = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $[\mathbf{S}] = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

2.60 Do the previous problem for the following matrix: $[\mathbf{T}] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$.

2.61 A tensor **T** has a matrix given below. Find the principal values and three mutually perpendicular principal directions.

$$[\mathbf{T}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

PART C: TENSOR CALCULUS

2.26 TENSOR-VALUED FUNCTIONS OF A SCALAR

Let $\mathbf{T} = \mathbf{T}(t)$ be a tensor-valued function of a scalar t (such as time). The derivative of **T** with respect to t is defined to be a second-order tensor given by:

$$\frac{d\mathbf{T}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)}{\Delta t}. \tag{2.26.1}$$

The following identities can be easily established:

$$\frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}, \tag{2.26.2}$$

$$\frac{d}{dt}(\alpha(t)\mathbf{T}) = \frac{d\alpha}{dt}\mathbf{T} + \alpha\frac{d\mathbf{T}}{dt}, \tag{2.26.3}$$

$$\frac{d}{dt}(\mathbf{TS}) = \frac{d\mathbf{T}}{dt}\mathbf{S} + \mathbf{T}\frac{d\mathbf{S}}{dt}, \tag{2.26.4}$$

$$\frac{d}{dt}(\mathbf{Ta}) = \frac{d\mathbf{T}}{dt}\mathbf{a} + \mathbf{T}\frac{d\mathbf{a}}{dt}, \tag{2.26.5}$$

$$\frac{d}{dt}(\mathbf{T}^{\mathrm{T}}) = \left(\frac{d\mathbf{T}}{dt}\right)^{\mathrm{T}}. \tag{2.26.6}$$

We shall prove here only Eq. (2.26.5). The other identities can be proven in a similar way. Using the definition given in Eq. (2.26.1), we have

$$\begin{aligned}
\frac{d}{dt}(\mathbf{Ta}) &= \lim_{\Delta t \to 0} \frac{\mathbf{T}(t+\Delta t)\mathbf{a}(t+\Delta t) - \mathbf{T}(t)\mathbf{a}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\mathbf{T}(t+\Delta t)\mathbf{a}(t+\Delta t) - \mathbf{T}(t)\mathbf{a}(t) - \mathbf{T}(t)\mathbf{a}(t+\Delta t) + \mathbf{T}(t)\mathbf{a}(t+\Delta t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\mathbf{T}(t+\Delta t)\mathbf{a}(t+\Delta t) - \mathbf{T}(t)\mathbf{a}(t+\Delta t) + \mathbf{T}(t)\mathbf{a}(t+\Delta t) - \mathbf{T}(t)\mathbf{a}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)}{\Delta t}\mathbf{a}(t+\Delta t) + \lim_{\Delta t \to 0} \mathbf{T}(t)\frac{\mathbf{a}(t+\Delta t) - \mathbf{a}(t)}{\Delta t}.
\end{aligned}$$

Thus,
$$\frac{d(\mathbf{Ta})}{dt} = \frac{d\mathbf{T}}{dt}\mathbf{a} + \mathbf{T}\frac{d\mathbf{a}}{dt}.$$

Example 2.26.1

Show that in Cartesian coordinates, the components of $d\mathbf{T}/dt$, i.e., $(d\mathbf{T}/dt)_{ij}$ are given by the derivatives of the components dT_{ij}/dt.

Solution

From

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j,$$

we have

$$\frac{dT_{ij}}{dt} = \frac{d\mathbf{e}_i}{dt} \cdot \mathbf{T}\mathbf{e}_j + \mathbf{e}_i \cdot \frac{d\mathbf{T}}{dt}\mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{T}\frac{d\mathbf{e}_j}{dt}.$$

Since the base vectors are fixed, their derivatives are zero; therefore,

$$\frac{dT_{ij}}{dt} = \mathbf{e}_i \cdot \frac{d\mathbf{T}}{dt}\mathbf{e}_j = \left(\frac{d\mathbf{T}}{dt}\right)_{ij}.$$

Example 2.26.2

Show that for an orthogonal tensor $\mathbf{Q}(t)$, $\left(\frac{d\mathbf{Q}}{dt}\right)\mathbf{Q}^{\mathrm{T}}$ is an antisymmetric tensor.

Solution

Since $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$, we have

$$\frac{d\left(\mathbf{Q}\mathbf{Q}^{\mathrm{T}}\right)}{dt} = \mathbf{Q}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} + \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} = \frac{d\mathbf{I}}{dt} = \mathbf{0}.$$

Since [see Eq. (2.26.6)] $\frac{d\mathbf{Q}^{T}}{dt} = \left(\frac{d\mathbf{Q}}{dt}\right)^{\mathrm{T}}$, therefore, the above equation leads to

$$\mathbf{Q}\left(\frac{d\mathbf{Q}}{dt}\right)^{\mathrm{T}} = -\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}.$$

Now $\quad \mathbf{Q}\left(\frac{d\mathbf{Q}}{dt}\right)^{\mathrm{T}} = \left(\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}}$; therefore,

$$\left(\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}} = -\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}},$$

that is, $\left(\frac{d\mathbf{Q}}{dt}\right)\mathbf{Q}^{\mathrm{T}}$ is an antisymmetric tensor.

Example 2.26.3

A time-dependent rigid body rotation about a fixed point can be represented by a rotation tensor $\mathbf{R}(t)$, so that a position vector $\mathbf{r}_{\mathrm{o}}$ is transformed through the rotation into $\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_{\mathrm{o}}$. Derive the equation

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}, \tag{2.26.7}$$

where $\boldsymbol{\omega}$ is the dual vector of the antisymmetric tensor $\frac{d\mathbf{R}}{dt}\mathbf{R}^{\mathrm{T}}$.

Solution

From $\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_o$, we obtain

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt}\mathbf{r}_o = \frac{d\mathbf{R}}{dt}\mathbf{R}^{-1}\mathbf{r} = \frac{d\mathbf{R}}{dt}\mathbf{R}^{\mathrm{T}}\mathbf{r}. \tag{i}$$

But $\frac{d\mathbf{R}}{dt}\mathbf{R}^{\mathrm{T}}$ is an antisymmetric tensor (see the previous example, Example 2.26.2) so that

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}, \tag{ii}$$

where $\boldsymbol{\omega}$ is the dual vector of $\frac{d\mathbf{R}}{dt}\mathbf{R}^{\mathrm{T}}$. From the well-known equation in rigid body kinematics, we can identify $\boldsymbol{\omega}$ as the angular velocity of the rigid body.

2.27 SCALAR FIELD AND GRADIENT OF A SCALAR FUNCTION

Let $\phi(\mathbf{r})$ be a scalar-valued function of the position vector $\mathbf{r}$. That is, for each position $\mathbf{r}$, $\phi(\mathbf{r})$ gives the value of a scalar, such as density, temperature, or electric potential at the point. In other words, $\phi(\mathbf{r})$ describes a scalar field. Associated with a scalar field is a vector field, called the *gradient* of ϕ. The gradient of ϕ at a point is defined to be a vector, denoted by grad ϕ or by $\nabla\phi$ such that its dot product with $d\mathbf{r}$ gives the difference of the values of the scalar at $\mathbf{r} + d\mathbf{r}$ and $\mathbf{r}$, i.e.,

$$d\phi = \phi(\mathbf{r} + d\mathbf{r}) - \phi(\mathbf{r}) = \nabla\phi \cdot d\mathbf{r}. \tag{2.27.1}$$

If dr denote the magnitude of $d\mathbf{r}$, and $\mathbf{e}$ the unit vector in the direction of $d\mathbf{r}$ (*Note:* $\mathbf{e} = d\mathbf{r}/dr$). Then the above equation gives, for $d\mathbf{r}$ in the $\mathbf{e}$ direction,

$$\frac{d\phi}{dr} = \nabla\phi \cdot \mathbf{e}. \tag{2.27.2}$$

That is, the component of $\nabla\phi$ in the direction of $\mathbf{e}$ gives the rate of change of ϕ in that direction (directional derivative). In particular, the components of $\nabla\phi$ in the coordinate directions $\mathbf{e}_i$ are given by

$$\frac{\partial\phi}{\partial x_i} = \left(\frac{d\phi}{dr}\right)_{\mathbf{e}_i-dir} = \nabla\phi \cdot \mathbf{e}_i. \tag{2.27.3}$$

Therefore, the Cartesian components of $\nabla\phi$ are $\partial\phi/\partial x_i$, that is,

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i. \tag{2.27.4}$$

The gradient vector has a simple geometrical interpretation. For example, if $\phi(\mathbf{r})$ describes a temperature field, then, on a surface of constant temperature (i.e., isothermal surface), $\phi =$ a constant. Let $\mathbf{r}$ be a point on an isothermal surface. Then, for any and all neighboring point $\mathbf{r}+d\mathbf{r}$ on the same isothermal surface, $d\phi = 0$. Thus,$\nabla\phi \cdot d\mathbf{r} = 0$. In other words, $\nabla\phi$ is a vector, perpendicular to the surface at the point $\mathbf{r}$. On the other hand, the dot product $\nabla\phi \cdot d\mathbf{r}$ is a maximum when $d\mathbf{r}$ is in the same direction as $\nabla\phi$. In other words, $\nabla\phi$ is greatest if $d\mathbf{r}$ is normal to the surface of constant ϕ and in this case, $d\phi = |\nabla\phi|dr$, or

$$\left(\frac{d\phi}{dr}\right)_{\max} = |\nabla\phi|, \tag{2.27.5}$$

for $d\mathbf{r}$ in the direction normal to the surface of constant temperature.

Example 2.27.1

If $\phi = x_1 x_2 + 2x_3$, find a unit vector **n** normal to the surface of a constant ϕ passing through the point (2,1,0).

Solution

By Eq. (2.27.4),

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 = x_2\mathbf{e}_1 + x_1\mathbf{e}_2 + 2\mathbf{e}_3.$$

At the point (2,1,0), $\nabla\phi = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3$. Thus,

$$\mathbf{n} = \frac{1}{3}(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3).$$

Example 2.27.2

If **q** denotes the heat flux vector (rate of heat transfer/area), the Fourier heat conduction law states that

$$\mathbf{q} = -k\nabla\Theta, \tag{i}$$

where Θ is the temperature field and k is thermal conductivity. If $\Theta = 2\left(x_1^2 + x_2^2\right)$, find $\nabla\Theta$ at the location A (1,0) and $B\left(1/\sqrt{2}, 1/\sqrt{2}\right)$. Sketch curves of constant Θ (isotherms) and indicate the vectors **q** at the two points.

Solution

By Eq. (2.27.4),

$$\nabla\Theta = \frac{\partial\Theta}{\partial x_1}\mathbf{e}_1 + \frac{\partial\Theta}{\partial x_2}\mathbf{e}_2 + \frac{\partial\Theta}{\partial x_3}\mathbf{e}_3 = 4x_1\mathbf{e}_1 + 4x_2\mathbf{e}_2.$$

Thus,

$$\mathbf{q} = -4k(x_1\mathbf{e}_1 + x_2\mathbf{e}_2).$$

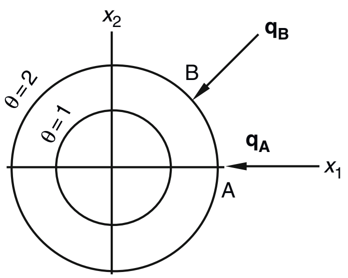

FIGURE 2.27-1

At point A,

$$\mathbf{q}_A = -4k\mathbf{e}_1,$$

and at point B,

$$\mathbf{q}_B = -2\sqrt{2}k(\mathbf{e}_1 + \mathbf{e}_2).$$

Clearly, the isotherms, Figure 2.27-1, are circles and the heat flux is an inward radial vector (consistent with heat flowing from higher to lower temperatures).

Example 2.27.3

A more general heat conduction law can be given in the following form:

$$\mathbf{q} = -\mathbf{K}\nabla\Theta,$$

where $\mathbf{K}$ is a tensor known as thermal conductivity tensor. (a) What tensor $\mathbf{K}$ corresponds to the Fourier heat conduction law mentioned in the previous example? (b) Find $\mathbf{q}$ if $\Theta = 2x_1 + 3x_2$, and

$$[\mathbf{K}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution

(a) Clearly, $\mathbf{K} = k\mathbf{I}$, so that $\mathbf{q} = -k\mathbf{I}\nabla\Theta = -k\nabla\Theta$.

(b) $\nabla\Theta = 2\mathbf{e}_1 + 3\mathbf{e}_2$ and

$$[\mathbf{q}] = -\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix}$$

that is,

$$\mathbf{q} = -\mathbf{e}_1 - 4\mathbf{e}_2,$$

which is clearly not normal to the isotherm (see Figure 2.27-2).

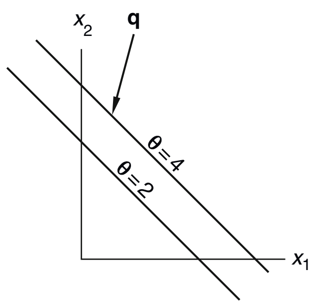

FIGURE 2.27-2

2.28 VECTOR FIELD AND GRADIENT OF A VECTOR FUNCTION

Let $\mathbf{v}(\mathbf{r})$ be a vector-valued function of position describing, for example, a displacement or a velocity field. Associated with $\mathbf{v}(\mathbf{r})$, is a tensor field, called the *gradient* of v, which is of considerable importance. The gradient of $\mathbf{v}$ (denoted by $\nabla\mathbf{v}$ or grad $\mathbf{v}$) is defined to be the second-order tensor, which, when operating on $d\mathbf{r}$, gives the difference of $\mathbf{v}$ at $\mathbf{r} + d\mathbf{r}$ and $\mathbf{r}$. That is,

$$d\mathbf{v} = \mathbf{v}(\mathbf{r} + d\mathbf{r}) - \mathbf{v}(\mathbf{r}) = (\nabla\mathbf{v})d\mathbf{r}. \tag{2.28.1}$$

Again, let dr denote $|d\mathbf{r}|$ and $\mathbf{e}$ denote $d\mathbf{r}/dr$; we have

$$\left(\frac{d\mathbf{v}}{dr}\right)_{\text{in } \mathbf{e}-\text{direction}} = (\nabla\mathbf{v})\mathbf{e}. \tag{2.28.2}$$

Therefore, the second-order tensor $\nabla\mathbf{v}$ transforms a unit vector $\mathbf{e}$ into the vector describing the rate of change of $\mathbf{v}$ in that direction. In Cartesian coordinates,

$$\left(\frac{d\mathbf{v}}{dr}\right)_{\text{in } \mathbf{e}_j-\text{direction}} = \frac{\partial\mathbf{v}}{\partial x_j} = (\nabla\mathbf{v})\mathbf{e}_j, \tag{2.28.3}$$

therefore, the components of $\nabla\mathbf{v}$ in indicial notation are given by

$$(\nabla\mathbf{v})_{ij} = \mathbf{e}_i \cdot (\nabla\mathbf{v})\mathbf{e}_j = \mathbf{e}_i \cdot \frac{\partial\mathbf{v}}{\partial x_j} = \frac{\partial(\mathbf{v}\cdot\mathbf{e}_i)}{\partial x_j} = \frac{\partial v_i}{\partial x_j}, \tag{2.28.4}$$

and in matrix form

$$[\nabla\mathbf{v}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}. \tag{2.28.5}$$

Geometrical interpretation of $\nabla\mathbf{v}$ will be given later in connection with the deformation of a continuum (Chapter 3).

2.29 DIVERGENCE OF A VECTOR FIELD AND DIVERGENCE OF A TENSOR FIELD

Let $\mathbf{v}(\mathbf{r})$ be a vector field. The *divergence* of $\mathbf{v}(\mathbf{r})$ is defined to be a scalar field given by the trace of the gradient of $\mathbf{v}$. That is,

$$\text{div } \mathbf{v} \equiv \text{tr}(\nabla\mathbf{v}). \tag{2.29.1}$$

In Cartesian coordinates, this gives

$$\text{div } \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_i}{\partial x_i}. \tag{2.29.2}$$

Let $\mathbf{T}(\mathbf{r})$ be a tensor field. The divergence of $\mathbf{T}(\mathbf{r})$ is defined to be a vector field, denoted by div $\mathbf{T}$, such that for any vector $\mathbf{a}$

$$(\text{div }\mathbf{T}) \cdot \mathbf{a} \equiv \text{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{a}\right) - \text{tr}\left(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{a}\right). \tag{2.29.3}$$

To find the Cartesian components of the vector div $\mathbf{T}$, let $\mathbf{b} = \text{div }\mathbf{T}$, then (*Note:* $\nabla\mathbf{e}_i = \mathbf{0}$ for Cartesian coordinates), from (2.29.3), we have

$$b_i = \mathbf{b} \cdot \mathbf{e}_i = \text{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{e}_i\right) - \text{tr}\left(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_i\right) = \text{div}\left(T_{ij}\mathbf{e}_j\right) - 0 = \partial T_{ij}/\partial x_j. \tag{2.29.4}$$

In other words,

$$\text{div }\mathbf{T} = \left(\partial T_{ij}/\partial x_j\right)\mathbf{e}_i. \qquad (2.29.5)^*$$

Example 2.29.1

Let $\alpha = \alpha(\mathbf{r})$ and $\mathbf{a} = \mathbf{a}(\mathbf{r})$. Show that $\text{div}(\alpha\mathbf{a}) = \alpha\text{div }\mathbf{a} + (\nabla\mathbf{v}) \cdot \mathbf{a}$.

Solution

Let $\mathbf{b} = \alpha\mathbf{a}$. Then $b_i = \alpha a_i$, so

$$\text{div }\mathbf{b} = \frac{\partial b_i}{\partial x_i} = \alpha\frac{\partial a_i}{\partial x_i} + \frac{\partial \alpha}{\partial x_i}a_i.$$

That is,

$$\text{div }(\alpha\mathbf{a}) = \alpha \text{ div }\mathbf{a} + (\nabla\alpha) \cdot \mathbf{a}. \tag{2.29.6}$$

Example 2.29.2

Given $\alpha = \alpha(\mathbf{r})$ and $\mathbf{T} = \mathbf{T}(\mathbf{r})$, show that

$$\text{div }(\alpha\mathbf{T}) = \mathbf{T}(\nabla\alpha) + \alpha \text{ div }\mathbf{T}. \tag{2.29.7}$$

Solution

We have, from (2.29.5),

$$\text{div }(\alpha\mathbf{T}) = \frac{\partial\left(\alpha T_{ij}\right)}{\partial x_j}\mathbf{e}_i = \frac{\partial\alpha}{\partial x_j}T_{ij}\mathbf{e}_i + \alpha\frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i = \mathbf{T}(\nabla\alpha) + \alpha \text{ div }\mathbf{T}.$$

---

*We note that the Cartesian components of the third-order tensor $\mathbf{M} \equiv \nabla\mathbf{T} = \nabla\left(T_{ij}\mathbf{e}_i\mathbf{e}_j\right)$are $\partial T_{ij}/\partial x_k$. In terms of $\mathbf{M} = M_{ijk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$, div $\mathbf{T}$ is a vector given by $M_{ijj}\mathbf{e}_i$. More on the components of $\nabla\mathbf{T}$ will be given in Chapter 8.

2.30 CURL OF A VECTOR FIELD

Let $\mathbf{v}(\mathbf{r})$ be a vector field. The *curl* of $\mathbf{v}(\mathbf{r})$ is defined to be a vector field given by twice the dual vector of the antisymmetric part of $\nabla\mathbf{v}$. That is

$$\text{curl } \mathbf{v} \equiv 2\mathbf{t}^{\mathbf{A}}, \tag{2.30.1}$$

where $\mathbf{t}^{\mathrm{A}}$ is the dual vector of $(\nabla\mathbf{v})^{\mathrm{A}}$.

In rectangular Cartesian coordinates,

$$[\nabla\mathbf{v}]^{\mathrm{A}} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) & 0 \end{bmatrix}. \tag{2.30.2}$$

Thus, the curl of $\mathbf{v}(\mathbf{r})$ is given by [see Eq. (2.21.3)]:

$$\text{curl } \mathbf{v} = 2\mathbf{t}^{\mathrm{A}} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3. \tag{2.30.3}$$

It can be easily verified that in indicial notation

$$\text{curl } \mathbf{v} = -\varepsilon_{ijk}\frac{\partial v_j}{\partial x_k}\mathbf{e}_i. \tag{2.30.4}$$

2.31 LAPLACIAN OF A SCALAR FIELD

Let $f(\mathbf{r})$ be a scalar-valued function of the position vector $\mathbf{r}$. The definition of the Laplacian of a scalar field is given by

$$\nabla^2 f = \text{div } (\nabla f) = \text{tr}(\nabla(\nabla f)). \tag{2.31.1}$$

In rectangular coordinates the Laplacian becomes

$$\nabla^2 f = \text{tr}(\nabla(\nabla f)) = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}. \tag{2.31.2}$$

2.32 LAPLACIAN OF A VECTOR FIELD

Let $\mathbf{v}(\mathbf{r})$ be a vector field. The Laplacian of $\mathbf{v}$ is defined by the following:

$$\nabla^2\mathbf{v} = \nabla\ (\text{div } \mathbf{v}) - \text{curl } (\text{curl } \mathbf{v}). \tag{2.32.1}$$

In rectangular coordinates,

$$\nabla(\text{div } \mathbf{v}) = \frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right)\mathbf{e}_i, \quad \text{curl } \mathbf{v} = -\varepsilon_{\alpha jk}\left(\frac{\partial v_j}{\partial x_k}\right)\mathbf{e}_\alpha, \tag{2.32.2}$$

and

$$\text{curl (curl } \mathbf{v}) = -\varepsilon_{i\alpha\beta}\frac{\partial}{\partial x_\beta}\left(-\varepsilon_{\alpha jk}\frac{\partial v_j}{\partial x_k}\right)\mathbf{e}_i = \varepsilon_{i\alpha\beta}\varepsilon_{\alpha jk}\frac{\partial}{\partial x_\beta}\left(\frac{\partial v_j}{\partial x_k}\right)\mathbf{e}_i. \tag{2.32.3}$$

Now $\varepsilon_{i\alpha\beta}\varepsilon_{\alpha jk} = -\varepsilon_{\alpha i\beta}\varepsilon_{\alpha jk} = -\left(\delta_{ij}\delta_{\beta k} - \delta_{ik}\delta_{\beta j}\right)$ [see Prob. 2.12], therefore,

$$\text{curl (curl } \mathbf{v}) = -\left(\delta_{ij}\delta_{\beta k} - \delta_{ik}\delta_{\beta j}\right)\frac{\partial}{\partial x_\beta}\left(\frac{\partial v_j}{\partial x_k}\right)\mathbf{e}_i = \left\{-\frac{\partial}{\partial x_\beta}\left(\frac{\partial v_i}{\partial x_\beta}\right) + \frac{\partial}{\partial x_\beta}\left(\frac{\partial v_\beta}{\partial x_i}\right)\right\}\mathbf{e}_i. \tag{2.32.4}$$

Thus,

$$\nabla^2\mathbf{v} = \nabla\ (\text{div } \mathbf{v}) - \text{curl (curl } \mathbf{v}) = \frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right)\mathbf{e}_i - \left\{-\frac{\partial}{\partial x_\beta}\left(\frac{\partial v_i}{\partial x_\beta}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial v_\beta}{\partial x_\beta}\right)\right\}\mathbf{e}_i. \tag{2.32.5}$$

That is, in rectangular coordinates,

$$\nabla^2\mathbf{v} = \frac{\partial^2 v_i}{\partial x_\beta \partial x_\beta}\mathbf{e}_i = \nabla^2 v_i \mathbf{e}_\text{i}. \tag{2.32.6}$$

In long form,

$$\nabla^2\mathbf{v} = \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right)\mathbf{e}_1 + \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2}\right)\mathbf{e}_2 + \left(\frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2}\right)\mathbf{e}_3. \tag{2.32.7}$$

Expressions for the polar, cylindrical, and spherical coordinate systems are given in Part D.

PROBLEMS FOR PART C

2.62 Prove the identity $\dfrac{d}{dt}(\mathbf{T} + \mathbf{S}) = \dfrac{d\mathbf{T}}{dt} + \dfrac{d\mathbf{S}}{dt}$ using the definition of derivative of a tensor.

2.63 Prove the identity $\dfrac{d}{dt}(\mathbf{TS}) = \mathbf{T}\dfrac{d\mathbf{S}}{dt} + \dfrac{d\mathbf{T}}{dt}\mathbf{S}$ using the definition of derivative of a tensor.

2.64 Prove that $\dfrac{d\mathbf{T}^\text{T}}{dt} = \left(\dfrac{d\mathbf{T}}{dt}\right)^\text{T}$ by differentiating the definition $\mathbf{a} \cdot \mathbf{Tb} = \mathbf{b} \cdot \mathbf{T}^\text{T}\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant vectors.

2.65 Consider the scalar field $\phi = x_1^2 + 3x_1x_2 + 2x_3$. (a) Find the unit vector normal to the surface of constant ϕ at the origin and at (1,0,1). (b) What is the maximum value of the directional derivative of ϕ at the origin? at (1,0,1)? (c) Evaluate $d\phi/dr$ at the origin if $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_3)$.

2.66 Consider the ellipsoidal surface defined by the equation $x^2/a^2 + y^2/b^2 + z^2/b^2 = 1$. Find the unit vector normal to the surface at a given point (x, y, z).

2.67 Consider the temperature field given by $\Theta = 3x_1x_2$. (a) If $\mathbf{q} = -k\nabla\Theta$, find the heat flux at the point $A(1,1,1)$. (b) If $\mathbf{q} = -\mathbf{K}\nabla\Theta$, find the heat flux at the same point, where

$$[\mathbf{K}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}.$$

2.68 Let $\phi(x_1, x_2, x_3)$ and $\psi(x_1, x_2, x_3)$ be scalar fields, and let $\mathbf{v}\ (x_1, x_2, x_3)$ and $\mathbf{w}\ (x_1, x_2, x_3)$ be vector fields. By writing the subscripted components form, verify the following identities:

(a) $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$, sample solution:

$$[\nabla(\phi + \psi)]_i = \frac{\partial(\phi + \psi)}{\partial x_i} = \frac{\partial \phi}{\partial x_i} + \frac{\partial \psi}{\partial x_i} = \nabla\phi + \nabla\psi,$$

(b) $\mathrm{div}(\mathbf{v} + \mathbf{w}) = \mathrm{div}\ \mathbf{v} + \mathrm{div}\ \mathbf{w}$, (c) $\mathrm{div}(\phi\mathbf{v}) = (\nabla\phi)\mathbf{v} + \phi(\mathrm{div}\ \mathbf{v})$ and (d) $\mathrm{div}(\mathrm{curl}\,\mathbf{v}) = 0$.

2.69 Consider the vector field $\mathbf{v} = x_1^2\mathbf{e}_1 + x_3^2\mathbf{e}_2 + x_2^2\mathbf{e}_3$. For the point (1,1,0), find (a) $\nabla\mathbf{v}$, (b) $(\nabla\mathbf{v})\mathbf{v}$, (c) div **v** and curl **v**, and (d) the differential $d\mathbf{v}$ for $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$.

PART D: CURVILINEAR COORDINATES

In Part C, the Cartesian components for various vector and tensor operations such as the gradient, the divergence, and the Laplacian of a scalar field and tensor fields were derived. In this part, components in polar, cylindrical, and spherical coordinates for these same operations will be derived.

2.33 POLAR COORDINATES

Consider polar coordinates (r,θ), (see Figure 2.33-1) such that

$$r = \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{x_2}{x_1}. \tag{2.33.1}$$

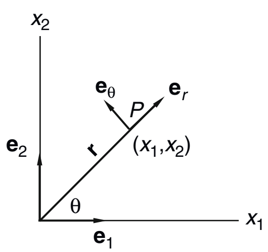

FIGURE 2.33-1

The unit base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ can be expressed in terms of the Cartesian base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as

$$\mathbf{e}_r = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2. \tag{2.33.2}$$

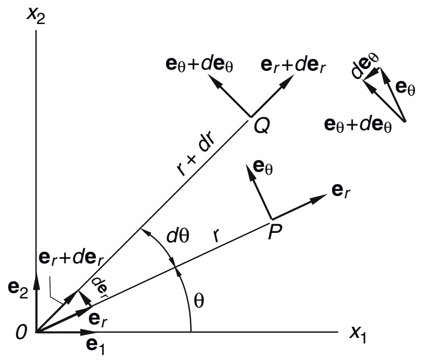

FIGURE 2.33-2

These unit base vectors vary in direction as θ changes. In fact, from Eqs. (2.33.2), we have

$$d\mathbf{e}_r = (-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2)d\theta = d\theta\mathbf{e}_\theta, \quad d\mathbf{e}_\theta = (-\cos\theta\mathbf{e}_1 - \sin\theta\mathbf{e}_2)d\theta = -d\theta\mathbf{e}_r. \tag{2.33.3}$$

The geometrical representation of $d\mathbf{e}_r$ and $d\mathbf{e}_\theta$ are shown in Figure 2.33-2, where one notes that $\mathbf{e}_r(P)$ has rotated an infinitesimal angle $d\theta$ to become $\mathbf{e}_r(Q) = \mathbf{e}_r(P) + d\mathbf{e}_r$ where $d\mathbf{e}_r$ is perpendicular to $\mathbf{e}_r(P)$ with a magnitude $|d\mathbf{e}_r| = (1)d\theta = d\theta$. Similarly, $d\mathbf{e}_\theta$ is perpendicular to $\mathbf{e}_\theta\,(P)$ but pointing in the negative $\mathbf{e}_r$ direction, and its magnitude is also $d\theta$.

Now, from the position vector

$$\mathbf{r} = r\mathbf{e}_r, \tag{2.33.4}$$

we have

$$d\mathbf{r} = dr\mathbf{e}_r + rd\mathbf{e}_r. \tag{2.33.5}$$

Using Eq. (2.33.3), we get

$$d\mathbf{r} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta. \tag{2.33.6}$$

The geometrical representation of this equation is also easily seen if one notes that $d\mathbf{r}$ is the vector PQ in the preceding figure.

The components of ∇f, $\nabla\mathbf{v}$, div $\mathbf{v}$, div $\mathbf{T}$, $\nabla^2 f$ and $\nabla^2\mathbf{v}$ in polar coordinates will now be obtained.

(i) Components of ∇f:

Let $f(r,\theta)$ be a scalar field. By definition of the gradient of f, we have

$$df = \nabla f \cdot d\mathbf{r} = (a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta)\cdot(dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta) = a_r dr + a_\theta r d\theta, \tag{2.33.7}$$

where a_r and a_θ are components of ∇f in the $\mathbf{e}_r$ and $\mathbf{e}_\theta$ direction, respectively. But from calculus,

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta. \tag{2.33.8}$$

Since Eqs. (2.33.7) and (2.33.8) must yield the same result for all increments $dr, d\theta$, we have

$$a_r = \frac{\partial f}{\partial r}, \quad a_\theta = \frac{1}{r}\frac{\partial f}{\partial\theta}, \tag{2.33.9}$$

thus,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta. \tag{2.33.10}$$

(ii) Components of $\nabla \mathbf{v}$: Let

$$\mathbf{v}(r,\theta) = v_r(r,\theta)\mathbf{e}_r + \mathrm{v}_\theta(r,\theta)\mathbf{e}_\theta. \tag{2.33.11}$$

By definition of $\nabla \mathbf{v}$, we have

$$d\mathbf{v} = \nabla \mathbf{v} d\mathbf{r}. \tag{2.33.12}$$

Let $\mathbf{T} = \nabla \mathbf{v}$. Then

$$d\mathbf{v} = \mathbf{T}d\mathbf{r} = \mathbf{T}(dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta) = dr\mathbf{T}\mathbf{e}_r + rd\theta\mathbf{T}\mathbf{e}_\theta. \tag{2.33.13}$$

Now

$$\mathbf{T}\mathbf{e}_r = T_{rr}\mathbf{e}_r + T_{\theta r}\mathbf{e}_\theta \quad \text{and} \quad \mathbf{T}\mathbf{e}_\theta = T_{r\theta}\mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta, \tag{2.33.14}$$

therefore,

$$d\mathbf{v} = (T_{rr}dr + T_{r\theta}rd\theta)\mathbf{e}_r + (T_{\theta r}dr + T_{\theta\theta}rd\theta)\mathbf{e}_\theta. \tag{2.33.15}$$

From Eq. (2.33.11), we also have

$$d\mathbf{v} = dv_r\mathbf{e}_r + v_r d\mathbf{e}_r + d\mathrm{v}_\theta\mathbf{e}_\theta + \mathrm{v}_\theta d\mathbf{e}_\theta. \tag{2.33.16}$$

Since [see Eq. (2.33.3)]

$$d\mathbf{e}_r = d\theta\mathbf{e}_\theta, \quad d\mathbf{e}_\theta = -d\theta\mathbf{e}_r, \tag{2.33.17}$$

therefore, Eq. (2.33.16) becomes

$$d\mathbf{v} = (dv_r - \mathrm{v}_\theta d\theta)\mathbf{e}_r + (v_r d\theta + d\mathrm{v}_\theta)\mathbf{e}_\theta. \tag{2.33.18}$$

From calculus,

$$dv_r = \frac{\partial v_r}{\partial r}dr + \frac{\partial v_r}{\partial \theta}d\theta, \quad dv_\theta = \frac{\partial v_\theta}{\partial r}dr + \frac{\partial v_\theta}{\partial \theta}d\theta. \tag{2.33.19}$$

Substituting Eq. (2.33.19) into Eq. (2.33.18), we have

$$d\mathbf{v} = \left[\frac{\partial v_r}{\partial r}dr + \left(\frac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right)d\theta\right]\mathbf{e}_r + \left[\frac{\partial v_\theta}{\partial r}dr + \left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)d\theta\right]\mathbf{e}_\theta. \tag{2.33.20}$$

Eq. (2.33.15) and Eq. (2.33.20), then, give

$$\frac{\partial v_r}{\partial r}dr + \left(\frac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right)d\theta = T_{rr}dr + T_{r\theta}rd\theta, \quad \frac{\partial v_\theta}{\partial r}dr + \left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)d\theta = T_{\theta r}dr + T_{\theta\theta}rd\theta. \tag{2.33.21}$$

Eq. (2.33.21) must hold for any values of dr and $d\theta$. Thus,

$$T_{rr} = \frac{\partial v_r}{\partial r}, \quad T_{r\theta} = \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right), \quad T_{\theta r} = \frac{\partial v_\theta}{\partial r}, \quad T_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right). \tag{2.33.22}$$

In matrix form,

$$[\nabla \mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right) \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) \end{bmatrix}. \tag{2.33.23}$$

(iii) div **v**:
Using the components of $\nabla \mathbf{v}$ given in (ii), that is, Eq. (2.33.23), we have

$$\text{div } \mathbf{v} = \text{tr}(\nabla \mathbf{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right). \tag{2.33.24}$$

(iv) Components of curl **v:**
The antisymmetric part of $\nabla \mathbf{v}$ is

$$[\nabla \mathbf{v}]^{\mathrm{A}} = \frac{1}{2}\begin{bmatrix} 0 & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right) - \dfrac{\partial v_\theta}{\partial r} \\ -\left\{\dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right) - \dfrac{\partial v_\theta}{\partial r}\right\} & 0 \end{bmatrix}. \tag{2.33.25}$$

Therefore, from the definition that curl $\mathbf{v}$ = twice the dual vector of $(\nabla \mathbf{v})^{\mathrm{A}}$, we have

$$\text{curl } \mathbf{v} = \left(\frac{\partial v_\theta}{\partial r} + \frac{\mathrm{v}_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_3. \tag{2.33.26}$$

(v) Components of div **T:**
The invariant definition of the divergence of a second-order tensor is

$$(\text{div } \mathbf{T}) \cdot \mathbf{a} = \text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{a}) - \text{tr}((\nabla \mathbf{a})\mathbf{T}^{\mathrm{T}}) \text{ for any } \mathbf{a}. \tag{2.33.27}$$

Take $\mathbf{a} = \mathbf{e}_r$; then the preceding equation gives

$$(\text{div } \mathbf{T})_r = \text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_r) - \text{tr}((\nabla \mathbf{e}_r)\mathbf{T}^{\mathrm{T}}). \tag{2.33.28}$$

To evaluate the first term on the right-hand side, we note that

$$\mathbf{T}^{\mathrm{T}}\mathbf{e}_r = T_{rr}\mathbf{e}_r + T_{r\theta}\mathbf{e}_\theta, \tag{2.33.29}$$

so that according to Eq. (2.33.24),

$$\text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_r) = \text{div}(T_{rr}\mathbf{e}_r + T_{r\theta}\mathbf{e}_\theta) = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\left(\frac{\partial T_{r\theta}}{\partial \theta} + T_{rr}\right). \tag{2.33.30}$$

To evaluate the second term, we first use Eq. (2.33.23) to obtain $\nabla \mathbf{e}_r$. In fact, since $\mathbf{e}_r = (1)\mathbf{e}_r + (0)\mathbf{e}_\theta$, we have, with $v_r = 1$ and $v_\theta = 0$,

$$[\nabla \mathbf{e}_r] = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{r} \end{bmatrix}, \quad [\nabla \mathbf{e}_r][\mathbf{T}^{\mathrm{T}}] = \begin{bmatrix} 0 & 0 \\ \dfrac{T_{r\theta}}{r} & \dfrac{T_{\theta\theta}}{r} \end{bmatrix}, \quad \mathrm{tr}([\nabla \mathbf{e}_r][\mathbf{T}^{\mathrm{T}}]) = \frac{T_{\theta\theta}}{r}. \tag{2.33.31}$$

Thus, Eq. (2.33.28) gives

$$(\mathrm{div}\ \mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r}. \tag{2.33.32}$$

In a similar manner, one can derive

$$(\mathrm{div}\ \mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r}. \tag{2.33.33}$$

(vi) Laplacian of $f(\mathbf{x})$:
Given a scalar field $f(\mathbf{x})$, the Laplacian of $f(\mathbf{x})$ is given by $\nabla^2 f = \mathrm{div}(\nabla f) = \mathrm{tr}(\nabla(\nabla f))$. In polar coordinates,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta. \tag{2.33.34}$$

From, $\mathrm{div}\ \mathbf{v} = \dfrac{\partial v_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_r}{r}$, we have

$$\nabla^2 f = \mathrm{div}\ \nabla f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}. \tag{2.33.35}$$

(vii) Laplacian of a vector field $\mathbf{v}(\mathbf{x})$:
Laplacian of $\mathbf{v}$ is given by: $\nabla^2 \mathbf{v} = \nabla(\mathrm{div}\ \mathbf{v}) - \mathrm{curl}\,\mathrm{curl}\ \mathbf{v}$. Now, in polar coordinates:

$$\begin{aligned} \nabla(\mathrm{div}\ \mathbf{v}) &= \frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\mathbf{e}_\theta \\ &= \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial^2 v_\theta}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2}\right)\mathbf{e}_r + \left(\frac{1}{r}\frac{\partial^2 v_r}{\partial \theta \partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_\theta, \end{aligned} \tag{2.33.36}$$

and

$$\mathrm{curl}\ \mathbf{v} = \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z. \tag{2.33.37}$$

Since [see Eq. (2.34.7)]

$$\mathrm{curl}\ \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z,$$

therefore,

$$(\mathrm{curl\ curl}\ \mathbf{v})_r = \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) = \left(\frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta \partial r} + \frac{1}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}\right), \tag{2.33.38}$$

$$(\text{curl curl } \mathbf{v})_\theta = -\frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) = \left(-\frac{\partial^2 v_\theta}{\partial r^2} - \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r^2} + \frac{1}{r}\frac{\partial^2 v_r}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial v_r}{\partial \theta}\right). \tag{2.33.39}$$

Thus,

$$(\nabla^2 \mathbf{v})_r = \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2}, \tag{2.33.40}$$

and

$$(\nabla^2 \mathbf{v})_\theta = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}. \tag{2.33.41}$$

2.34 CYLINDRICAL COORDINATES

In cylindrical coordinates, the position of a point P is determined by (r, θ, z), where r and θ determine the position of the vertical projection of the point P on the xy plane (the point P' in Figure 2.34-1) and the coordinate z determines the height of the point P from the xy plane. In other words, the cylindrical coordinates is a polar coordinate (r, θ) in the xy plane plus a coordinate z perpendicular to the xy plane.

We shall denote the position vector of P by $\mathbf{R}$, rather than $\mathbf{r}$, to avoid confusion between the position vector $\mathbf{R}$ and the coordinate r (which is a radial distance in the xy plane). The unit vector $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are on the xy plane and it is clear from Figure 2.34-1 that

$$\mathbf{R} = r\mathbf{e}_r + \mathrm{z}\mathbf{e}_z, \tag{2.34.1}$$

and

$$d\mathbf{R} = dr\mathbf{e}_r + rd\mathbf{e}_r + d\mathrm{z}\mathbf{e}_z + \mathrm{z}d\mathbf{e}_z. \tag{2.34.2}$$

In the preceding equation, $d\mathbf{e}_r$ is given by exactly the same equation given earlier for the polar coordinates [Eq. (2.33.3)]. We note also that $\mathbf{e}_z$ never changes its direction or magnitude regardless where the point P is, thus $d\mathbf{e}_z = 0$. Therefore,

$$d\mathbf{R} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + d\mathrm{z}\mathbf{e}_z. \tag{2.34.3}$$

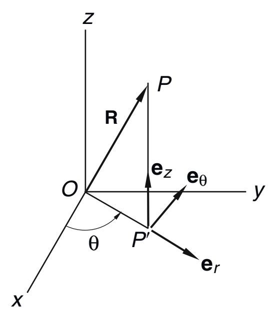

FIGURE 2.34-1

By retracing all the steps used in the previous section on polar coordinates, we can easily obtain the following results:

(i) Components of ∇f:

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z. \tag{2.34.4}$$

(ii) Components of $\nabla \mathbf{v}$:

$$[\nabla \mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix}. \tag{2.34.5}$$

(iii) div $\mathbf{v}$:

$$\text{div } \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z}. \tag{2.34.6}$$

(iv) Components of curl $\mathbf{v}$:

The vector curl $\mathbf{v}$ = twice the dual vector of $(\nabla \mathbf{v})^{\mathrm{A}}$, thus,

$$\text{curl } \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{\partial v_\theta}{\partial r} + \frac{\mathrm{v}_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z. \tag{2.34.7}$$

(v) Components of div $\mathbf{T}$:

$$(\text{div}\,\mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z}, \tag{2.34.8}$$

$$(\text{div}\,\mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r} + \frac{\partial T_{\theta z}}{\partial z}, \tag{2.34.9}$$

$$(\text{div}\,\mathbf{T})_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r}. \tag{2.34.10}$$

(vi) Laplacian of f:

$$\nabla^2 f = \text{div } \nabla f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial z^2}. \tag{2.34.11}$$

(vii) Laplacian of $\mathbf{v}$:

$$(\nabla^2 \mathbf{v})_r = \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}, \tag{2.34.12}$$

$$(\nabla^2 \mathbf{v})_\theta = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}, \quad (2.34.13)$$

$$(\nabla^2 \mathbf{v})_z = \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}. \quad (2.34.14)$$

2.35 SPHERICAL COORDINATES

In Figure 2.35-1, we show the spherical coordinates (r, θ, ϕ) of a general point P. In this figure, $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors in the direction of increasing r, θ and ϕ, respectively.

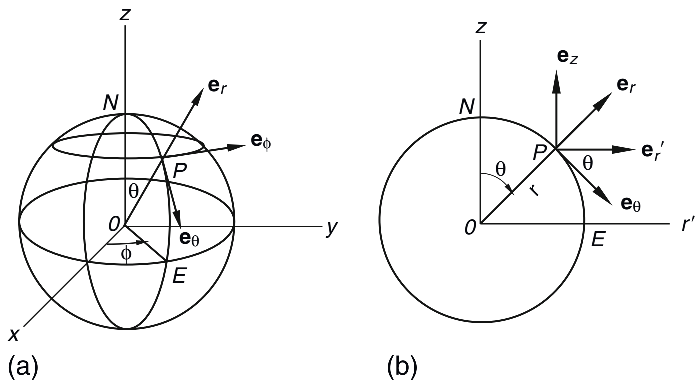

FIGURE 2.35-1

The position vector for the point P can be written as

$$\mathbf{r} = r\mathbf{e}_r, \quad (2.35.1)$$

where r is the magnitude of the vector $\mathbf{r}$. Thus,

$$d\mathbf{r} = dr\mathbf{e}_r + rd\mathbf{e}_r. \quad (2.35.2)$$

To evaluate $d\mathbf{e}_r$ we note from Figure 2.35-1(b) that

$$\mathbf{e}_r = \cos\theta\,\mathbf{e}_z + \sin\theta\,\mathbf{e}'_r, \quad \mathbf{e}_\theta = \cos\theta\,\mathbf{e}'_r - \sin\theta\,\mathbf{e}_z, \quad (2.35.3)$$

where $\mathbf{e}'_r$ is the unit vector in the OE (i.e., r') direction (r' is in the xy plane). Thus,

$$d\mathbf{e}_r = -\sin\theta\,d\theta\,\mathbf{e}_z + \cos\theta\,d\mathbf{e}_z + \cos\theta\,d\theta\,\mathbf{e}'_r + \sin\theta\,d\mathbf{e}'_r = \left(-\sin\theta\,\mathbf{e}_z + \cos\theta\,\mathbf{e}'_r\right)d\theta + \sin\theta\,d\mathbf{e}'_r,$$

that is,

$$d\mathbf{e}_r = d\theta\,\mathbf{e}_\theta + \sin\theta\,d\mathbf{e}'_r. \quad (2.35.4)$$

Now, just as in polar coordinates, due to $d\phi$,

$$d\mathbf{e}'_r = d\phi\mathbf{e}_\phi, \quad (2.35.5)$$

therefore,

$$d\mathbf{e}_r = d\theta\mathbf{e}_\theta + \sin\theta d\phi\mathbf{e}_\phi. \tag{2.35.6}$$

Now, from the second equation of (2.35.3), we have,

$$d\mathbf{e}_\theta = -\sin\theta d\theta\mathbf{e}'_r + \cos\theta d\mathbf{e}'_r - \cos\theta d\theta\mathbf{e}_z = -(\sin\theta\mathbf{e}'_r + \cos\theta\mathbf{e}_z)d\theta + \cos\theta d\mathbf{e}'_r.$$

Using Eq. (2.35.3) and Eq. (2.35.5), the preceding equation becomes

$$d\mathbf{e}_\theta = -\mathbf{e}_r d\theta + \cos\theta d\phi\mathbf{e}_\phi. \tag{2.35.7}$$

From Figure 2.35-1(a) and similar to the polar coordinate, we have

$$d\mathbf{e}_\phi = d\phi(-\mathbf{e}'_r). \tag{2.35.8}$$

With $\mathbf{e}'_r = \cos\theta\mathbf{e}_\theta + \sin\theta\mathbf{e}_r$ (see Figure 2.35-1(b)), the preceding equation becomes

$$d\mathbf{e}_\phi = -\sin\theta d\phi\mathbf{e}_r - \cos\theta d\phi\mathbf{e}_\theta. \tag{2.35.9}$$

Summarizing the preceding, we have

$$d\mathbf{e}_r = d\theta\mathbf{e}_\theta + \sin\theta d\phi\mathbf{e}_\phi, \quad d\mathbf{e}_\theta = -\mathbf{e}_r d\theta + \cos\theta d\phi\mathbf{e}_\phi, \quad d\mathbf{e}_\phi = -\sin\theta d\phi\mathbf{e}_r - \cos\theta d\phi\mathbf{e}_\theta, \tag{2.35.10}$$

and from Eq. (2.35.2), we have

$$d\mathbf{r} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{e}_\phi. \tag{2.35.11}$$

We can now obtain the components of ∇f, $\nabla\mathbf{v}$, div $\mathbf{v}$, curl $\mathbf{v}$, div $\mathbf{T}$, $\nabla^2 f$, and $\nabla^2\mathbf{v}$ for spherical coordinates.

(i) Components of ∇f:
Let $f(r,\theta,\phi)$ be a scalar field. By the definition of ∇f, we have

$$df = \nabla f \cdot d\mathbf{r} = \left[(\nabla f)_r\mathbf{e}_r + (\nabla f)_\theta\mathbf{e}_\theta + (\nabla f)_\phi\mathbf{e}_\phi\right] \cdot (dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{e}_\phi), \tag{2.35.12}$$

that is,

$$df = (\nabla f)_r\, dr + (\nabla f)_\theta r d\theta + (\nabla f)_\phi r\sin\theta d\phi. \tag{2.35.13}$$

From calculus, the total derivative of df is

$$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial\theta}d\theta + \frac{\partial f}{\partial\phi}d\phi. \tag{2.35.14}$$

Comparing Eq. (2.35.14) and Eq. (2.35.13), we have

$$(\nabla f)_r = \frac{\partial f}{\partial r}, \quad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial\theta}, \quad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}. \tag{2.35.15}$$

(ii) Components of $\nabla\mathbf{v}$:
Let the vector field be represented by

$$\mathbf{v}(r,\theta,\phi) = v_r(r,\theta,\phi)\mathbf{e}_r + v_\theta(r,\theta,\phi)\mathbf{e}_\theta + v_\phi(r,\theta,\phi)\mathbf{e}_\phi. \tag{2.35.16}$$

Letting $\mathbf{T} = \nabla \mathbf{v}$, we have

$$d\mathbf{v} = \mathbf{T}d\mathbf{r} = \mathbf{T}\left(dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{e}_\phi\right) = dr\mathbf{T}\mathbf{e}_r + rd\theta\mathbf{T}\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{T}\mathbf{e}_\phi. \tag{2.35.17}$$

By the definition of components of a tensor T in spherical coordinates, we have

$$\begin{aligned}\mathbf{T}\mathbf{e}_r &= T_{rr}\mathbf{e}_r + T_{\theta r}\mathbf{e}_\theta + T_{\phi r}\mathbf{e}_\phi,\\ \mathbf{T}\mathbf{e}_\theta &= T_{r\theta}\mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta + T_{\phi\theta}\mathbf{e}_\phi,\\ \mathbf{T}\mathbf{e}_\phi &= T_{r\phi}\mathbf{e}_r + T_{\theta\phi}\mathbf{e}_\theta + T_{\phi\phi}\mathbf{e}_\phi.\end{aligned} \tag{2.35.18}$$

Substituting these into Eq. (2.35.17), we get

$$\begin{aligned}d\mathbf{v} = &\left(T_{rr}dr + T_{r\theta}rd\theta + T_{r\phi}r\ \sin\theta d\phi\right)\mathbf{e}_r + \left(T_{\theta\theta}rd\theta + T_{\theta r}dr + T_{\theta\phi}r\ \sin\theta d\phi\right)\mathbf{e}_\theta\\ &+\left(T_{\phi r}dr + T_{\phi\theta}rd\theta + T_{\phi\phi}r\ \sin\theta d\phi\right)\mathbf{e}_\phi.\end{aligned} \tag{2.35.19}$$

We also have, from Eq. (2.35.16),

$$d\mathbf{v} = dv_r\mathbf{e}_r + v_r d\mathbf{e}_r + dv_\theta\mathbf{e}_\theta + v_\theta d\mathbf{e}_\theta + dv_\phi\mathbf{e}_\phi + v_\phi d\mathbf{e}_\phi. \tag{2.35.20}$$

Using the expression for the total derivatives:

$$\begin{aligned}dv_r &= \frac{\partial v_r}{\partial r}dr + \frac{\partial v_r}{\partial\theta}d\theta + \frac{\partial v_r}{\partial\phi}d\phi,\\ dv_\theta &= \frac{\partial v_\theta}{\partial r}dr + \frac{\partial v_\theta}{\partial\theta}d\theta + \frac{\partial v_\theta}{\partial\phi}d\phi,\\ dv_\phi &= \frac{\partial v_\phi}{\partial r}dr + \frac{\partial v_\phi}{\partial\theta}d\theta + \frac{\partial v_\phi}{\partial\phi}d\phi,\end{aligned} \tag{2.35.21}$$

Eq. (2.35.10) and Eq. (2.35.20) become

$$\begin{aligned}d\mathbf{v} = &\left\{\frac{\partial v_r}{\partial r}dr + \left(\frac{\partial v_r}{\partial\theta} - v_\theta\right)d\theta + \left(\frac{\partial v_r}{\partial\phi} - v_\phi\sin\theta\right)d\phi\right\}\mathbf{e}_r\\ &+\left\{\frac{\partial v_\theta}{\partial r}dr + \left(v_r + \frac{\partial v_\theta}{\partial\theta}\right)d\theta + \left(\frac{\partial v_\theta}{\partial\phi} - v_\phi\ \cos\theta\right)d\phi\right\}\mathbf{e}_\theta\\ &+\left\{\frac{\partial v_\phi}{\partial r}dr + \frac{\partial v_\phi}{\partial\theta}d\theta + \left(\frac{\partial v_\phi}{\partial\phi} + v_r\ \sin\theta + v_\theta\ \cos\theta\right)d\phi\right\}\mathbf{e}_\phi.\end{aligned} \tag{2.35.22}$$

Now, comparing Eq. (2.35.22) with Eq. (2.35.19), we have

$$\begin{aligned}\left(T_{rr}dr + T_{r\theta}rd\theta + T_{r\phi}r\ \sin\theta d\phi\right) &= \left\{\frac{\partial v_r}{\partial r}dr + \left(\frac{\partial v_r}{\partial\theta} - v_\theta\right)d\theta + \left(\frac{\partial v_r}{\partial\phi} - v_\phi\ \sin\theta\right)d\phi\right\},\\ \left(T_{\theta r}dr + T_{\theta\theta}rd\theta + T_{\theta\phi}r\ \sin\theta d\phi\right) &= \left\{\frac{\partial v_\theta}{\partial r}dr + \left(v_r + \frac{\partial v_\theta}{\partial\theta}\right)d\theta + \left(\frac{\partial v_\theta}{\partial\phi} - v_\phi\ \cos\theta\right)d\phi\right\},\\ \left(T_{\phi r}dr + T_{\phi\theta}rd\theta + T_{\phi\phi}r\ \sin\theta d\phi\right) &= \left\{\frac{\partial v_\phi}{\partial r}dr + \frac{\partial v_\phi}{\partial\theta}d\theta + \left(\frac{\partial v_\phi}{\partial\phi} + v_r\ \sin\theta + v_\theta\ \cos\theta\right)d\phi\right\}.\end{aligned} \tag{2.35.23}$$

These equations must be valid for arbitrary values of dr, $d\theta$ and $d\phi$, therefore,

$$
\begin{aligned}
&T_{rr} = \frac{\partial v_r}{\partial r}, \quad T_{r\theta} r = \left(\frac{\partial v_r}{\partial \theta} - v_\theta\right), \quad T_{r\phi} r \sin\theta = \left(\frac{\partial v_r}{\partial \phi} - v_\phi \sin\theta\right), \\
&T_{\theta r} = \frac{\partial v_\theta}{\partial r}, \quad T_{\theta\theta} r = \left(v_r + \frac{\partial v_\theta}{\partial \theta}\right), \quad T_{\theta\phi} r \sin\theta = \left(\frac{\partial v_\theta}{\partial \phi} - v_\phi \cos\theta\right), \\
&T_{\phi r} = \frac{\partial v_\phi}{\partial r}, \quad T_{\phi\theta} r = \frac{\partial v_\phi}{\partial \theta}, \quad T_{\phi\phi} r \sin\theta = \left(\frac{\partial v_\phi}{\partial \phi} + v_r \sin\theta + v_\theta \cos\theta\right).
\end{aligned}
\tag{2.35.24}
$$

In matrix form, we have

$$
[\nabla \mathbf{v}] = \begin{bmatrix}
\frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r} \\
\frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi \cot\theta}{r} \\
\frac{\partial v_\phi}{\partial r} & \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}
\end{bmatrix}.
\tag{2.35.25}
$$

(iii) div $\mathbf{v}$:
Using Eq. (2.35.25), we obtain

$$
\begin{aligned}
\text{div } \mathbf{v} = \text{tr}(\nabla \mathbf{v}) &= \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{2v_r}{r} + \frac{v_\theta \cot\theta}{r} \\
&= \frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.
\end{aligned}
\tag{2.35.26}
$$

(iv) Components of curl $\mathbf{v}$:
The vector curl $\mathbf{v}$ = twice the dual vector of $(\nabla \mathbf{v})^A$, therefore

$$
\begin{aligned}
\text{curl } \mathbf{v} = &\left\{\frac{v_\phi \cot\theta}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right\}\mathbf{e}_r + \left\{\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial (r v_\phi)}{\partial r}\right\}\mathbf{e}_\theta \\
&+ \left\{\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right\}\mathbf{e}_\phi.
\end{aligned}
\tag{2.35.27}
$$

(v) Components of div $\mathbf{T}$:
Using the definition of div $\mathbf{T}$ given in Eq. (2.33.27) and take $\mathbf{a} = \mathbf{e}_r$, we have

$$
(\text{div } \mathbf{T})_r = \text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_r) - \text{tr}((\nabla \mathbf{e}_r)\mathbf{T}^{\mathrm{T}}).
\tag{2.35.28}
$$

To evaluate the first term on the right-hand side, we note that

$$
\mathbf{T}^{\mathrm{T}}\mathbf{e}_r = T_{rr}\mathbf{e}_r + T_{r\theta}\mathbf{e}_\theta + T_{r\phi}\mathbf{e}_\phi,
\tag{2.35.29}
$$

so that by using Eq. (2.35.26) for the divergence of a vector in spherical coordinates, we obtain,

$$\operatorname{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_r) = \frac{1}{r^2}\frac{\partial(r^2 T_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{r\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}. \tag{2.35.30}$$

To evaluate the second term in Eq. (2.35.28), we first used Eq. (2.35.25) to evaluate $\nabla\mathbf{e}_r$, then calculate $(\nabla\mathbf{e}_r)\mathbf{T}^{\mathrm{T}}$:

$$[\nabla\mathbf{e}_r] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/r \end{bmatrix},\ [(\nabla\mathbf{e}_r)\mathbf{T}^{\mathrm{T}}] = \begin{bmatrix} 0 & 0 & 0 \\ T_{r\theta}/r & T_{\theta\theta}/r & T_{\phi\theta}/r \\ T_{r\phi}/r & T_{\theta\phi}/r & T_{\phi\phi}/r \end{bmatrix} \tag{2.35.31}$$

thus,

$$\operatorname{tr}((\nabla\mathbf{e}_r)\mathbf{T}^{\mathrm{T}}) = \frac{T_{\theta\theta}}{r} + \frac{T_{\phi\phi}}{r}. \tag{2.35.32}$$

Substituting Eq. (2.35.32) and Eq. (2.35.30) into Eq. (2.35.28), we obtain,

$$(\operatorname{div}\mathbf{T})_r = \frac{1}{r^2}\frac{\partial(r^2 T_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{r\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi} - \frac{T_{\theta\theta}+T_{\phi\phi}}{r}. \tag{2.35.33}$$

In a similar manner, we can obtain (see Prob. 2.75)

$$(\operatorname{div}\mathbf{T})_\theta = \frac{1}{r^3}\frac{\partial(r^3 T_{\theta r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{T_{r\theta}-T_{\theta r}-T_{\phi\phi}\cot\theta}{r} \tag{2.35.34}$$

$$(\operatorname{div}\mathbf{T})_\phi = \frac{1}{r^3}\frac{\partial(r^3 T_{\phi r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\phi\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{T_{r\phi}-T_{\phi r}+T_{\theta\phi}\cot\theta}{r}. \tag{2.35.35}$$

(vi) Laplacian of f:
From

$$\begin{aligned} \operatorname{div}\mathbf{v} &= \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}, \\ \nabla f &= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi, \end{aligned} \tag{2.35.36}$$

we have

$$\begin{aligned} \nabla^2 f = \operatorname{div}(\nabla f) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right) \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial\theta^2}\right) + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 f}{\partial\phi^2}\right). \end{aligned} \tag{2.35.37}$$

(vii) Laplacian of a vector function **v**:
It can be obtained (see Prob. 2.75)

$$\nabla(\text{div}\,\mathbf{v}) = \left(\frac{1}{r^2}\frac{\partial^2 r^2 v_r}{\partial r^2} - \frac{2}{r^3}\frac{\partial r^2 v_r}{\partial r} + \frac{1}{r\sin\theta}\left(\frac{\partial^2 v_\theta \sin\theta}{\partial r\partial\theta} + \frac{\partial^2 v_\phi}{\partial r\partial\phi}\right) - \frac{1}{r^2\sin\theta}\left(\frac{\partial v_\theta \sin\theta}{\partial\theta} + \frac{\partial v_\phi}{\partial\phi}\right)\right)\mathbf{e}_r$$
$$+\left(\frac{1}{r^3}\frac{\partial^2 r^2 v_r}{\partial\theta\partial r} + \frac{1}{r^2\sin\theta}\left(\frac{\partial^2 v_\theta \sin\theta}{\partial\theta^2} + \frac{\partial^2 v_\theta \sin\theta}{\partial\theta^2}\right) - \frac{1}{r^2}\left(\frac{\cos\theta}{\sin^2\theta}\right)\frac{\partial v_\theta \sin\theta}{\partial\theta} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_\theta$$
$$+\left(\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2(v_\theta\sin\theta)}{\partial\phi\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial^2\phi}\right)\mathbf{e}_\phi, \tag{2.35.38}$$

and

$$\text{curl curl}\,\mathbf{v} = \left\{\frac{1}{r^2}\left(\frac{\partial^2 r v_\theta}{\partial\theta\partial r} - \frac{\partial^2 v_r}{\partial\theta^2}\right) + \frac{\cot\theta}{r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right) - \left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} - \frac{1}{r^2\sin\theta}\frac{\partial^2 r v_\phi}{\partial\phi\partial r}\right)\right\}\mathbf{e}_r$$
$$+\left\{\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 v_\phi\sin\theta}{\partial\phi\partial\theta} - \frac{\partial^2 v_\theta}{\partial\phi^2}\right) - \frac{1}{r^2}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial\theta}\right) - \left(\frac{1}{r}\frac{\partial^2 r v_\theta}{\partial r^2} - \frac{1}{r^2}\frac{\partial r v_\theta}{\partial r} - \frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\right)\right\}\mathbf{e}_\theta$$
$$+\left\{\begin{array}{l}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial r\partial\phi} - \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial^2 r v_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial r v_\phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial r v_\phi}{\partial r}\right) \\ -\frac{1}{r^2}\frac{1}{\sin\theta}\left(-v_\phi\sin\theta + \sin\theta\frac{\partial^2 v_\phi}{\partial\theta^2} - \frac{\partial^2 v_\theta}{\partial\theta\partial\phi}\right) + \frac{\cos\theta}{r^2\sin^2\theta}\left(\frac{\partial v_\phi\sin\theta}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right)\end{array}\right\}\mathbf{e}_\phi. \tag{2.35.39}$$

Thus, $\nabla^2\mathbf{v} = \nabla(\text{div}\,\mathbf{v}) - \text{curl curl}\,\mathbf{v}$ leads to:

$$(\nabla^2\mathbf{v})_r = \left(\begin{array}{l}\frac{1}{r^2}\frac{\partial^2 r^2 v_r}{\partial r^2} - \frac{2}{r^3}\frac{\partial r^2 v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} \\ -\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\end{array}\right), \tag{2.35.40}$$

$$(\nabla^2\mathbf{v})_\theta = \left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right\} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} \\ +\frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\end{array}\right), \tag{2.35.41}$$

$$(\nabla^2\mathbf{v})_\phi = \left(\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right\} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} \\ +\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial v_\theta}{\partial\phi}\end{array}\right). \tag{2.35.42}$$

PROBLEMS FOR PART D

2.70 Calculate div **u** for the following vector field in cylindrical coordinates:

(a) $u_r = u_\theta = 0, \quad u_z = A + Br^2.$
(b) $u_r = \sin\theta/r, \quad u_\theta = u_z = 0.$
(c) $u_r = r^2 \sin\theta/2, \quad u_\theta = r^2\cos\theta/2, \quad u_z = 0.$

2.71 Calculate $\nabla\mathbf{u}$ for the following vector field in cylindrical coordinates:

$$u_r = A/r, \quad u_\theta = Br, \quad u_z = 0.$$

2.72 Calculate div **u** for the following vector field in spherical coordinates:

$$u_r = Ar + \frac{B}{r^2}, \quad u_\theta = u_\phi = 0.$$

2.73 Calculate $\nabla\mathbf{u}$ for the following vector field in spherical coordinates:

$$u_r = Ar + B/r^2, \quad u_\theta = u_\phi = 0.$$

2.74 From the definition of the Laplacian of a vector, $\nabla^2\mathbf{v} = \nabla(\text{div }\mathbf{v}) - \text{curl curl }\mathbf{v}$, derive the following results in cylindrical coordinates:

$$(\nabla^2\mathbf{v})_r = \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2}\right) \quad \text{and}$$

$$(\nabla^2\mathbf{v})_\theta = \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2}.$$

2.75 From the definition of the Laplacian of a vector, $\nabla^2\mathbf{v} = \nabla(\text{div }\mathbf{v}) - \text{curl curl }\mathbf{v}$, derive the following result in spherical coordinates:

$$(\nabla^2\mathbf{v})_r = \left(\frac{1}{r^2}\frac{\partial^2 r^2 v_r}{\partial r^2} - \frac{2}{r^3}\frac{\partial r^2 v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right).$$

2.76 From the equation $(\text{div }\mathbf{T})\cdot\mathbf{a} = \text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{a}) - \text{tr}(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{a})$ [see Eq. (2.29.3)], verify that in polar coordinates the θ-component of the vector $(\text{div }\mathbf{T})$ is:

$$(\text{div }\mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{r\theta} + T_{\theta r}}{r}.$$

2.77 Calculate div **T** for the following tensor field in cylindrical coordinates:

$$T_{rr} = A + \frac{B}{r^2}, \quad T_{\theta\theta} = A - \frac{B}{r^2}, \quad T_{zz} = \text{constant}, \quad T_{r\theta} = T_{\theta r} = T_{rz} = T_{zr} = T_{\theta z} = T_{z\theta} = 0.$$

2.78 Calculate div **T** for the following tensor field in cylindrical coordinates:

$$T_{rr} = \frac{Az}{R^3} - \frac{3Br^2 z}{R^5}, \quad T_{\theta\theta} = \frac{Az}{R^3}, \quad T_{zz} = -\left(\frac{Az}{R^3} + \frac{3Bz^3}{R^5}\right), \quad T_{rz} = T_{zr} = -\left(\frac{Ar}{R^3} + \frac{3Brz^2}{R^5}\right),$$
$$T_{r\theta} = T_{\theta r} = T_{\theta z} = T_{z\theta} = 0, \quad R^2 = r^2 + z^2.$$

2.79 Calculate div **T** for the following tensor field in spherical coordinates:

$$T_{rr} = A - \frac{2B}{r^3}, \quad T_{\theta\theta} = T_{\phi\phi} = A + \frac{B}{r^3}, \quad T_{r\theta} = T_{\theta r} = T_{\theta\phi} = T_{\phi\theta} = T_{r\phi} = T_{\phi r} = 0.$$

2.80 From the equation $(\text{div}\,\mathbf{T}) \cdot \mathbf{a} = \text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{a}) - \text{tr}(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{a})$ [see Eq. (2.29.3)], verify that in spherical coordinates the θ-component of the vector (div **T**) is:

$$(\text{div}\,\mathbf{T})_\theta = \frac{1}{r^3}\frac{\partial(r^3 T_{\theta r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{T_{r\theta} - T_{\theta r} - T_{\phi\phi}\cot\theta}{r}.$$