Network Algorithms and Complexity

Agreement in Unreliable Distributed Systems

Aris Pagourtzis, Dimitris Sakavalas

CoReLab, NTUA

Introduction

Distributed Computing
In an unreliable environment

- Several interacting entities (players/agents) that cooperate to achieve a common goal in the absence of a central authority.

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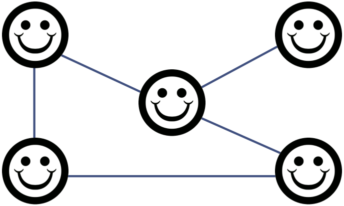

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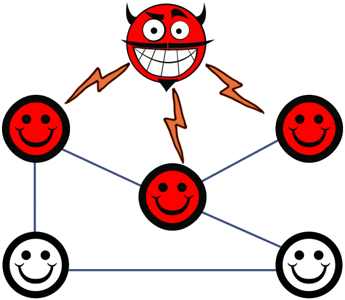

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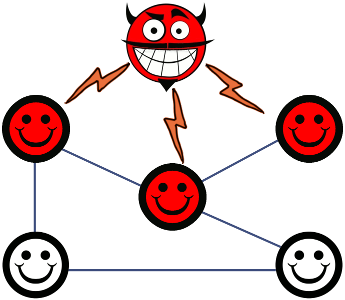

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- Players arranged in a communication network G.
- Central adversary corrupts/controls players and makes them misbehave (e.g. false messages, crash).
- Goal: Achieve common goal despite the presence of corruptions.

Agreement under corruptions

Two major variations of the problem [Lamport, Shostak, Pease, 1982]

Broadcast (Byzantine Generals)

The goal is to have some designated player, called the dealer, consistently send a message to all other players.

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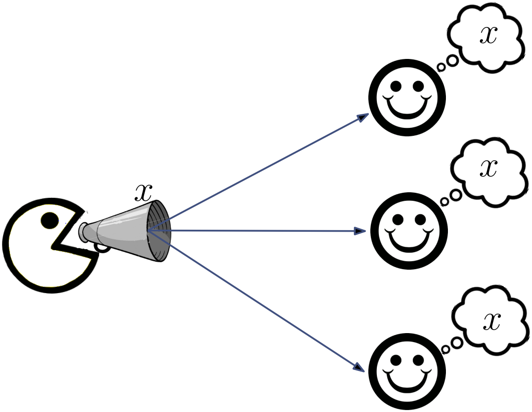
Consensus (Byzantine Agreement)

Goal: Make all players agree on the same output value given that every player starts with an input value.

If all correct players hold the same input value then the output value is required to be the same as this input value.

Ideal Broadcast



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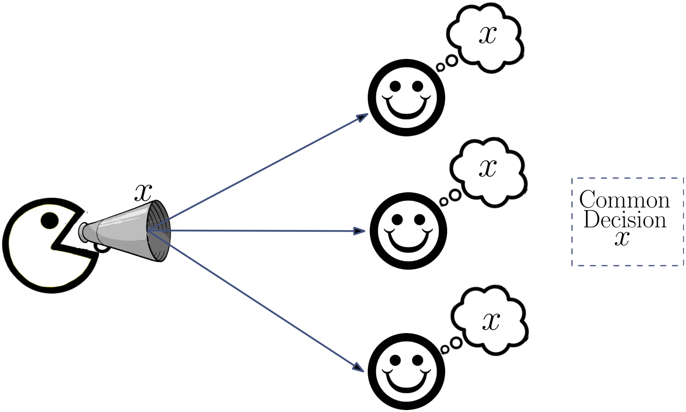

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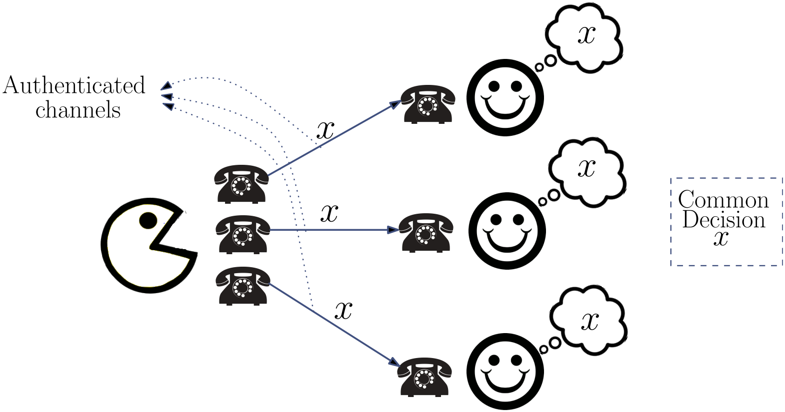
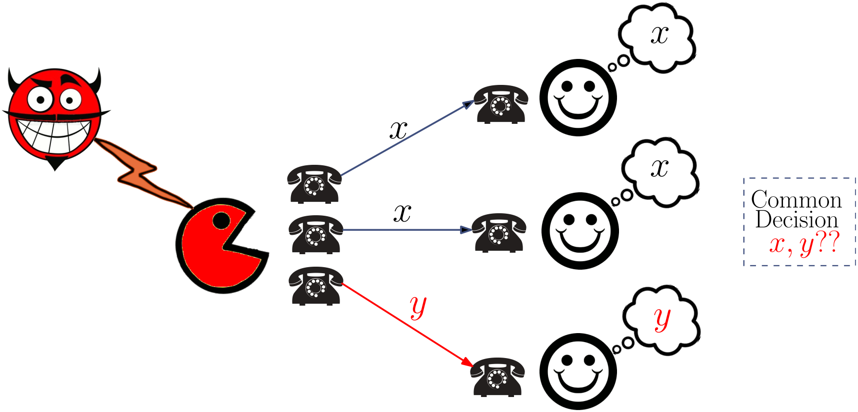
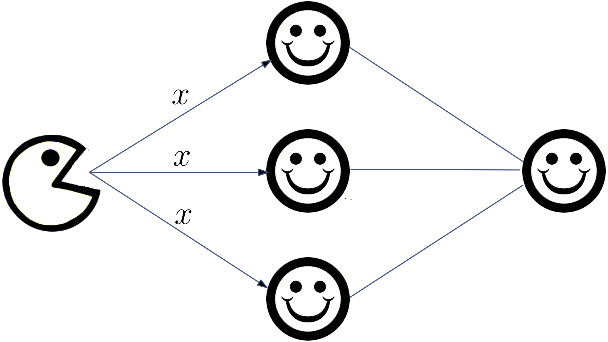


Real Broadcast with Corrupted Dealer



Broadcast in Incomplete Networks



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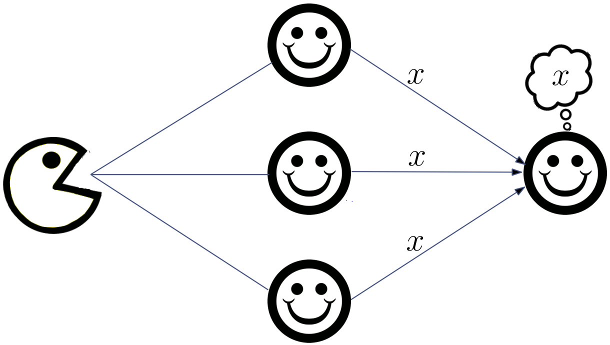

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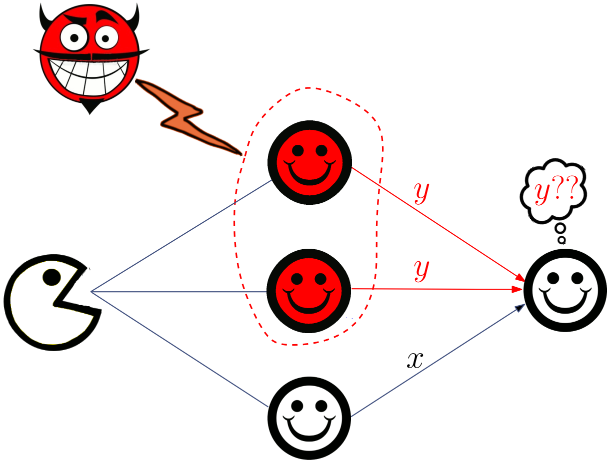

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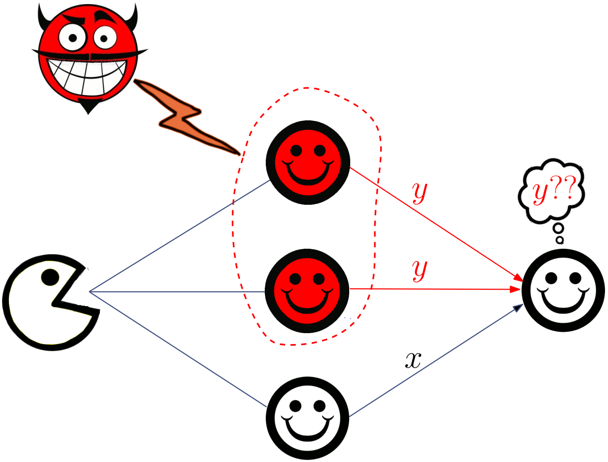

Even Broadcast with an honest dealer is non trivial in this case.

Problem Definition

Player Set: $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$, **Initial input space:** $\mathcal{X}$,
Corrupted players set:$\mathcal{T} \subseteq \mathcal{V}$, **Honest Players Set:** $\mathcal{H} = \mathcal{V} \setminus \mathcal{T}$
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Dealer $D \in \mathcal{V}$ with **input value** $\mathbf{x_D} \in \mathcal{X}$. Π is a Broadcast protocol for $\mathcal{V}$ if it satisfies:

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Consensus (Byzantine Agreement)

Every player $v \in \mathcal{V}$ has an **input value** $\mathbf{x_v} \in \mathcal{X}$. Π is a Consensus protocol for $\mathcal{V}$ if it satisfies:

1. **(Consistency)**
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   If all honest players have the same input value x then all honest players decide x.

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Extreme Corruption cases, e.g., Consensus with one honest player..

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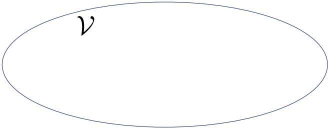

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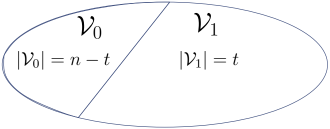

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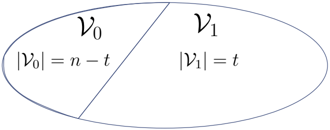

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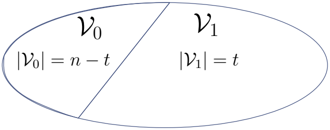

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Output

1. If all honest players output x and $\mathcal{T} = \mathcal{V}_x$ then validity is violated.
2. If honest players compute different outputs and $\mathcal{T} = \emptyset$ then consistency is violated. □

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Theorem.

If $t < n/2$ then (efficient) Broadcast is achievable iff (efficient) Consensus is achievable.

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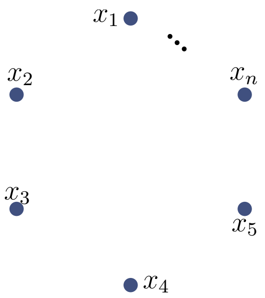

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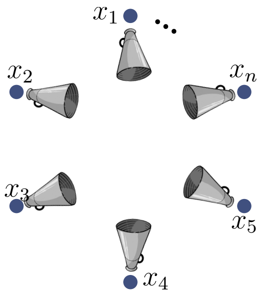

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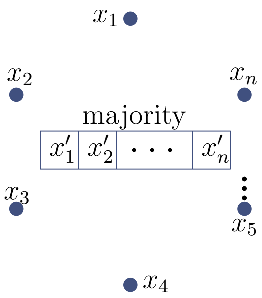

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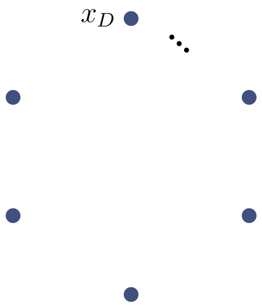

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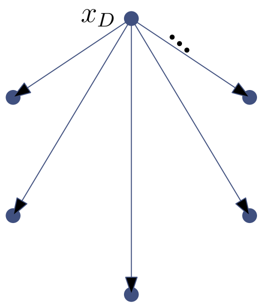

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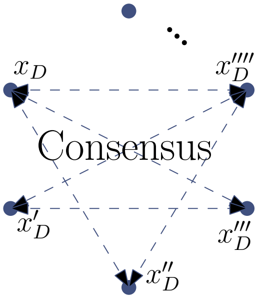

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  (to probabilistic polynomial time computations in a security parameter κ).

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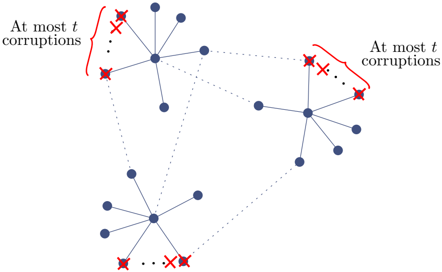

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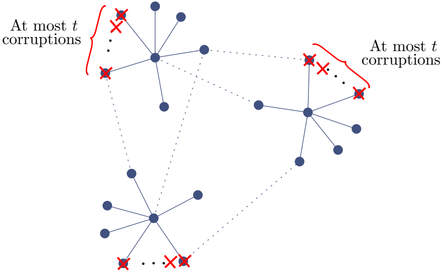

- General Adversary Model [HM97]: Monotone family (structure) $\mathcal{Z} \in 2^V$ of admissible corruption player-sets. Subsumes all other models.

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Asynchronous Model: Honest players cannot wait for messages from more than $n - t$ players in each round, where n is the number of players and t the number of corruptions tolerated.

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Consistently shared data: Typically a PKI.

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Synchronous round: One clock cycle of the global clock.
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Fully Polynomial Protocol

Protocol of polynomial Bit, Round and Local Computations Complexity.

Broadcast Protocols

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Improvement of trade-off between Resilience, BC, RC and LCC. local computation complexity.

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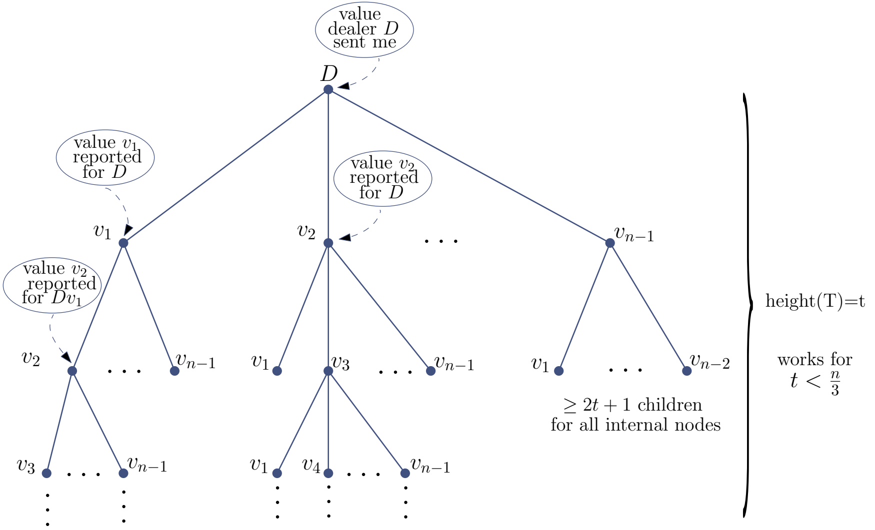
Protocol	n	RC	BC	LCC
[PSL80]	$3t+1$	$t+1$	$exp(n)$	$exp(n)$
[DFF+82]	$3t+1$	$2t+c$	$poly(n)$	$poly(n)$
[Coa86]	$4t+1$	$t+\frac{t}{d}$	$O(n^d)$	$\exp(n)$
[BNDDS92]	$3t+1$	$t+\frac{t}{d}$	$O(n^d)$	$O(n^d)$
[MW88]	$6t+1$	$t+1$	$poly(n)$	$poly(n)$
[BG93]	$4t+1$	$t+1$	$poly(n)$	$poly(n)$
[BG91]	$(3+\epsilon)t$	$t+1$	$poly(n)\cdot O(2^{1/\epsilon})$	$poly(n)\cdot O(2^{1/\epsilon})$
[GM98]	$3t+1$	$t+1$	$poly(n)$	$poly(n)$

Exponential Information Gathering

The EIG Tree



EIG Algorithm I - Information Gathering

Information Gathering

Round 1

1. Dealer sends its initial value x_D to the $n-1$ other players and decides on x_D.
2. Each v stores value x_D in the root of $tree_v$ ($tree_v(D) := x_D$). A special default value of $\perp$ is stored if the Dealer failed to send a legitimate value in X.

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Round h, $2 \leq h \leq t+1$

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EIG Algorithm I - Information Gathering

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Intuitively, v stores in node $D \ldots qr$ the value that "r says q says $\ldots$ the source said".

EIG Algorithm II - Data Conversion

After $t+1$ rounds o Information Gathering, each player v computes the commonly agreed-upon recursive function *resolve*() in order to decide.

Resolve Function

(Recursive majority of descendants of node a)
For all a sequences of $tree_v$:

$$resolve_v(a) = \begin{cases} tree(a) & \text{, if } a \text{ is a leaf;} \\ m & \text{, If } m \text{ is the majority of } resolve \text{ applied} \\ & \quad \text{to the children of } a; \\ \perp & \text{, If } a \text{ is not a leaf and no majority exists.} \end{cases}$$

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Decision

Player v decides on the value $resolve_v(D)$.

Complexity of the EIG Algorithm

Theorem (Lamport, Shostak, Pease 1982).

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For any $1 \leq h \leq t+1$, the h-round EIG tree has $O(n^{h-1})$ leaves, yielding messages of size $O(n^{h-1})$ in round $h+1$. Thus, BC and LCC grow exponential in t.

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[GM98]: First $(t+1)$-round fully polynomial, optimal resilience Broadcast protocol.

Reducing the Communication Cost

- 1989: P.Berman, J.Garay, K. Perry, first communication efficient 1/3−resilient protocol. Basis of many later protocols.
- *King Consensus* Protocol. Using the equivalence of Broadcast-Consensus easily transformed in a Broadcast protocol.
- Input value space $X = \{0, 1, \perp\}$(Binary Consensus). Can be used to achieve General Consensus with an overhead of 2 extra rounds and $O(n^2 \cdot b)$ extra bits, where b : maximum length of a message [Coa87].

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Protocol: $WeakConsensus(x_1 \dots x_n) \rightarrow (y_1 \dots y_n)$

1. Every $v_i \in \mathcal{V}$ sends x_i to all v_j.
   Let c_m^j be the copies of a message $m \in \{0, 1\}$ received by player v_j in this round.
2. Every v_j computes:
$$y_j = \begin{cases} m & \text{if } c_m^j \geq n - t \\ \perp & else \end{cases}$$
3. Every $v_j \in \mathcal{V}$ returns y_j

Weak Consensus Correctness

Lemma.

WeakConsensus achieves Weak Consistency and Validity for $t < n/3$.

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Proof.

Validity: Let $x_i = x$, $\forall v_i \in \mathcal{V}$.
Step 2: All $v_i \in \mathcal{H}$ collect the value x at least $n - t$ times, thus all $v_i \in \mathcal{H}$ receive the value $1 - x$ at most $t < n - t$ (since $t < n/3$) and they all decide on $y_i = x$.

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Weak Consistency: Let $v_i, v_j \in \mathcal{H}$ and $y_i = 0$. Thus $c_0^i \geq n-t$. That means that at least $n-2t$ honest players sent him this value.
Consequently

$$c_0^j \geq n-2t \Rightarrow c_1^j = n-n+2t = 2t < n-t$$

So v_j computes either $y_j = 0$ or $y_j = \perp$. □

Graded Consistency

Every $v_i \in \mathcal{V}$ computes y_i and the *grade value* $g_i \in \{0, 1\}$.

Graded Consistency

If $v_i \in \mathcal{H}$ decides on $y_i \in \{0, 1\}$ with $g_i = 1$ then every other $v_j \in \mathcal{H}$ decides on $y_j = y_i$.

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Protocol: $\mathit{GradedConsensus}(\mathbf{x_1}, \ldots, \mathbf{x_n}) \rightarrow ((\mathbf{y_1}, \mathbf{g_1}), \ldots, (\mathbf{y_n}, \mathbf{g_n}))$

1. $(z_1, \ldots, z_n) := \mathit{WeakConsensus}(x_1, \ldots, x_n)$
2. Every $v_i \in \mathcal{V}$ sends z_i to all v_j.
3. Every v_j computes:
$$y_j = \begin{cases} 1 & \text{if } c_1^j > c_0^j \\ 0 & \mathit{else} \end{cases}, \qquad g_j = \begin{cases} 1 & \text{if } c_{y_j}^j \geq n - t \\ 0 & \mathit{else} \end{cases}$$
4. Every $v_j \in \mathcal{V}$ returns (y_j, g_j)

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Validity: If $\forall\, v_i, v_j \in \mathcal{H},\ x_i = x$ then $(y_i, g_i) = (x, 1)$.
Let x be the common input value. After step 1, $z_i = x, \forall v_i \in \mathcal{H}$, due to WeakConsensus. Validity remains in a similar way as in WeakConsensus.

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Graded Consistency: Let $v_i, v_j \in \mathcal{H}$ and let v_i output $(y_i, 1)$.
That means that at least $n - 2t$ honest players sent him $z_k = y_i$.

Player v_j also receives y_i from $n - 2t$ honest players. The remaining $t + 1$ honest send him either y_i or $\perp$ due to WeakConsensus. Thus,

$$c^j_{1-y_i} \leq t < n - 2t \Rightarrow y_j = y_i$$

□

King Consistency

A player v_k is chosen to be the king.

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Protocol: $KingConsensus(v_k, x_1, \ldots x_n) \rightarrow (y_1, \ldots, y_n)$

1. $((z_1, g_1) \ldots, (z_n, g_n)) := GradedConsensus(x_1, \ldots, x_n)$
2. The king v_k sends z_k to all players.
3. Every v_j computes

$$y_j = \begin{cases} z_j & \text{if } g_j = 1 \\ z_k & \textit{else} \end{cases}$$

4. Every v_j returns y_j

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Validity: If $\forall\ v_i, v_j \in \mathcal{H},\ x_i = x$ then due to Graded Consistency of step 1 these players compute $(z_i, g_i) = (x, 1)$. Therefore Every $v_i \in \mathcal{H}$ outputs $y_i = x$.

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King Consistency: Let the king $v_k \in \mathcal{H}$

If $\forall\ v_i \in \mathcal{H},\ g_i = 0$ in step 1 then all honest v_i output $y_i = z_k$ in step 3.

If $\exists\ v_i \in \mathcal{H}$ with $g_i = 1$, because of Graded Consistency all honest (king included) computed the same z_i, thus they output $y_i = z_i$

□

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Consensus$(x_1, \ldots, x_n) \rightarrow (y_1, \ldots, y_n)$

1. For $k := 1$ *to* $t+1$
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Observation

If the king is honest, by King Consistency all honest players decide on the same output value v which will be their input value for the next round. Due to the fact that the KingConsensus sub-protocol maintains Validity the final decision value of each honest player will remain v.

Broadcast Protocol

Protocol: $Broadcast(x, D) \rightarrow (y_1 \ldots, y_n)$

1. Dealer D sends x to all players
2. $(y_1, \ldots, y_n) := Consensus(x_1, \ldots, x_n)$,
   with x_i the value that player v_i received from the Dealer.
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Theorem ([BG89]).

The above protocol achieves Broadcast (Consensus) with resiliency $n > 3t$, $BC = O(n^2 t)$ *and* $RC = 3t + O(1)$.

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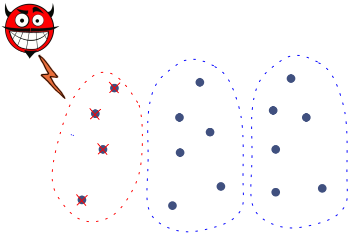
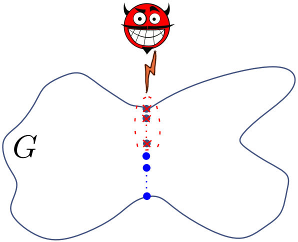
Proof. Each sub-protocol is executed $t + 1$ times and involves *one-to-all* bit communication for every player $BC = O(n^2 t)$
King Consensus: 3 rounds,one for each sub-protocol
$RC = 3t + O(1)$ □

Parameter Lower Bounds

Parameter Lower Bounds -Overview

- Resiliency: $n > 3t$ (Interactive Consistency) [PSL80]
- Bit Complexity: $BC \geq n(t+1)/4$ [DR85]
- Round Complexity: $RC \geq t+1$ [FL82, DS83]
- Connectivity of Network G: $conn(G) > 2t$ [Dol82]





Scenaria–Executions

- **State Assignment** C_i**:** An assignment of states to each player.
- **Message assignment** M_i**:** An assignment of a message to each channel.

A Scenario is defined to be an infinite sequence:

$$\sigma = C_0, M_1, C_1, M_2, C_2, \dots$$

Indistiguishable Scenaria ($\sigma \overset{v}{\sim} \sigma'$)

Two scenaria σ, σ' are indistinguishable with respect to player v, $\sigma \overset{v}{\sim} \sigma'$ if v has the same *view*(v), i.e., the same sequence of states, outgoing and incoming messages.

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decision(v): deterministic function of $\mathit{view}(v)$ (Perfect Security).

Connectivity Lower Bound $(conn(G) > 2t)$

σ_0	σ_1
$x_D = 0$ $T = C_0$	$x_D = 1$ $T = C_1$

Corrupted players C_i of scenario σ_i act like in σ_{1-i}.

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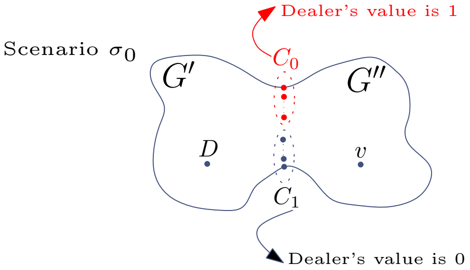

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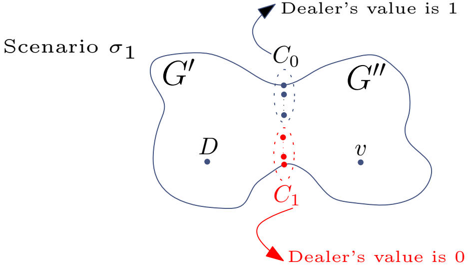

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Assume that v_0, v_1, v_2 solve Broadcast in two rounds given that $t = 1$:

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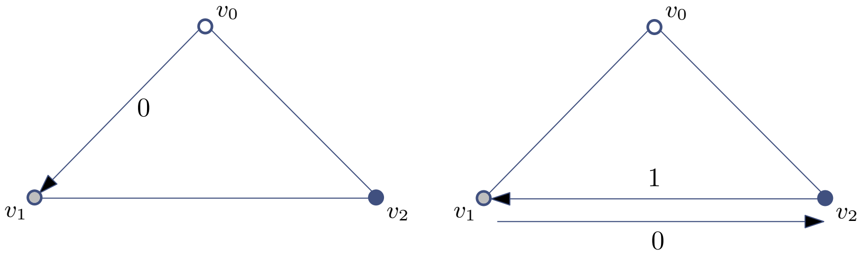
Honest player v_1, knowing that at most one of the v_0, v_2 is corrupted, has to decide on a value that satisfies both conditions of the Broadcast problem. Consider the following *view*(v_1).



Resiliency Lower Bound-Example

Two possible scenarios σ_1(corrupted v_2) and σ_2(corrupted v_0) s.t. $\sigma_1 \overset{v_1}{\sim} \sigma_2$ (indistinguishable with respect to v_1):

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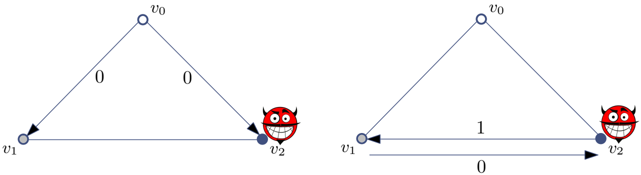

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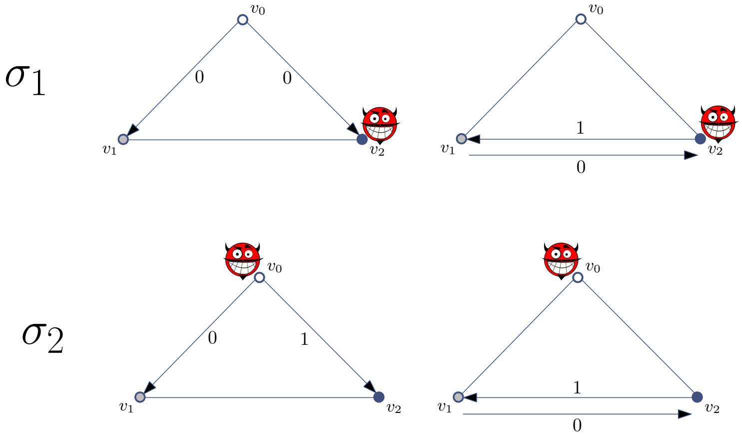

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If $decision(v_1) = 1$ and σ_1 holds, then validity is violated, thus

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$(1), (2) \Rightarrow$ Consistency is violated.

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The algorithm uses only two rounds and particular types of messages.

Resiliency Lower Bound I

Lemma 3.1.

Three players cannot solve the Broadcast problem in the presence of one fault ($n = 3$ and $t = 1$).

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Proof. Assume the existence of algorithm $\mathcal{A}$ that achieves Broadcast in system T in the presence of a corrupted player. Construct system H using two copies of T,

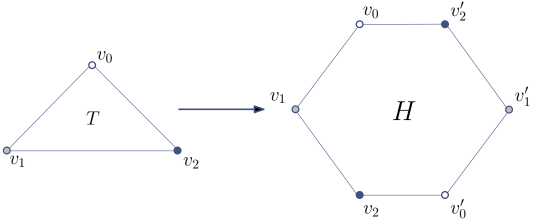

Figure : Identical copy $v'_k = v_{k+3}$ of v_k. Connect $v_{k \mod 6}$ with $v_{(k+1) \mod 6}$ and $v_{(k-1) \mod 6}$

Resiliency Lower Bound II

In H all players run $\mathcal{A}$ and have only local names for their neighbors.

Claim

For all σ_H scenario of H without adversary and $\forall k \in \{0, \ldots, 5\}$, $\exists \sigma_T$ scenario of T in which $v_{(k+2) \bmod 3}$ is corrupted s.t.

$$\sigma_H \overset{v_k}{\sim} \sigma_T \text{ and } \sigma_H \overset{v_{k+1 \bmod 6}}{\sim} \sigma_T$$

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For v_k and $v_{k+1 \bmod 6}$, their views are indistinguishable from their views as players $v_{k \bmod 3}$ and $v_{(k+1) \bmod 3}$ in T where the adversary corrupts $v_{(k+2) \bmod 3}$ by simply simulating all the remaining players of H.

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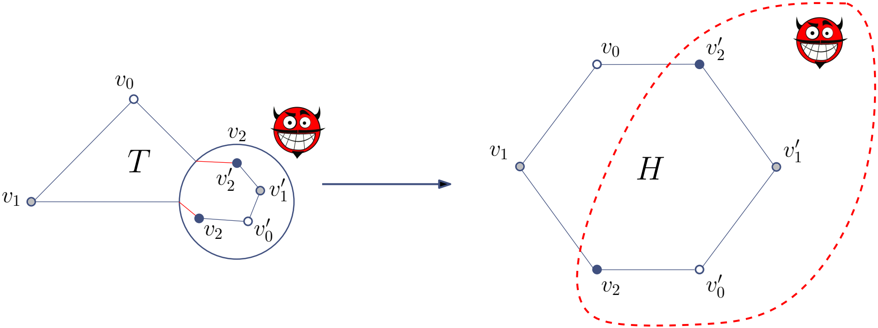
Thus, every such pair executes A in H without adversary and achieves Broadcast. If H exhibits contradictory behavior then A cannot exist.

Resiliency Lower Bound III

Example.

The adversary corrupts v_2 in T by simulating the subsystem of H encircled



Resiliency Lower Bound IV

Contradictory behavior of H

H involves two players v_0, v_0' of the type corresponding to the Dealer. Suppose they have inputs $x_0 \in \{0, 1\}$ and $x_0' = 1 - x_0$ respectively.

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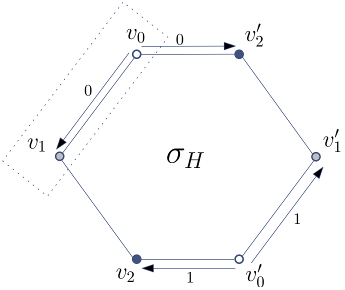
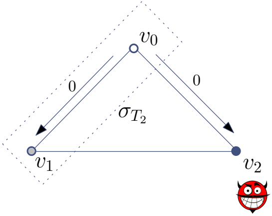
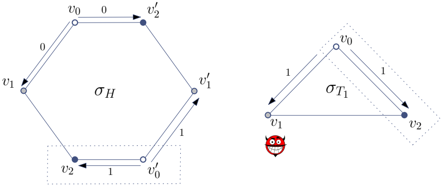
H involves two players v_0, v_0' of the type corresponding to the Dealer. Suppose they have inputs $x_0 \in \{0,1\}$ and $x_0' = 1 - x_0$ respectively.





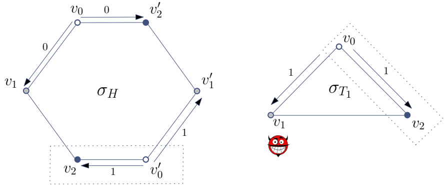
$$\sigma_H \overset{v_0}{\sim} \sigma_{T_2} \text{ and } \sigma_H \overset{v_1}{\sim} \sigma_{T_2} \Rightarrow \mathit{decision}(v_1) = 0 \quad (1)$$

Resiliency Lower Bound V

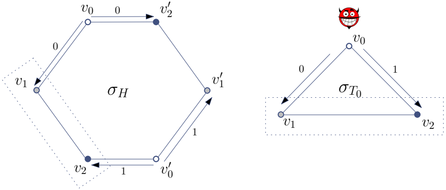


$$\sigma_H \overset{v_0'}{\sim} \sigma_{T_1} \text{ and } \sigma_H \overset{v_2}{\sim} \sigma_{T_1} \Rightarrow decision(v_2) = 1 \quad (2)$$

Resiliency Lower Bound V

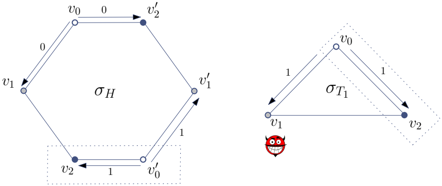


$$\sigma_H \overset{v_0'}{\sim} \sigma_{T_1} \text{ and } \sigma_H \overset{v_2}{\sim} \sigma_{T_1} \Rightarrow decision(v_2) = 1 \qquad (2)$$

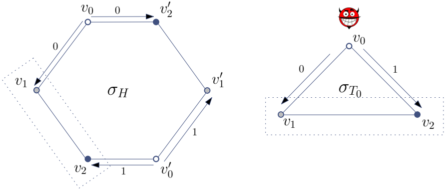


$$\sigma_H \overset{v_1}{\sim} \sigma_{T_0} \text{ and } \sigma_H \overset{v_2}{\sim} \sigma_{T_0} \Rightarrow decision(v_1) = decision(v_2) \qquad (3)$$

Resiliency Lower Bound V



$$\sigma_H \overset{v_0'}{\sim} \sigma_{T_1} \text{ and } \sigma_H \overset{v_2}{\sim} \sigma_{T_1} \Rightarrow \mathit{decision}(v_2) = 1 \qquad (2)$$



$$\sigma_H \overset{v_1}{\sim} \sigma_{T_0} \text{ and } \sigma_H \overset{v_2}{\sim} \sigma_{T_0} \Rightarrow \mathit{decision}(v_1) = \mathit{decision}(v_2) \qquad (3)$$

Relations (1), (2) and (3) yield a contradiction. □

Resiliency Lower Bound VI

Theorem 3.2.

There is no solution to the Broadcast problem for n players in the presence of t corrupted players, if $3 \leq n \leq 3t$

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Idea: Assume Broadcast protocol A with dealer v_0 for $|\mathcal{V}| = n, |T| \geq n/3$. Transform A into B Broadcast protocol for $|\mathcal{V}| = 3, |T| = 1$.

Let partition $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ s.t. $\forall i$, $1 \leq |\mathcal{V}_i| \leq t$. We let each v_i simulate every $v \in \mathcal{V}_i$ (messages and computation steps)

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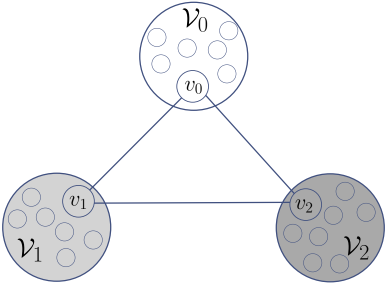
Protocol B

Player v_0: dealer in protocol B.
If in A: $v \in \mathcal{V}_i$ sends m to $u \in \mathcal{V}_j$, $i \neq j$, then
B: v_i sends m to v_j along with the identities of v, u.
If in A: $v \in \mathcal{V}_i$ decides on m, then
B: v_i decides on the value m. (If there are multiple values chooses one)

Resiliency Lower Bound VII



In A, $T_A = \mathcal{V}_j$, where $T_B = v_j$ ($|T_A| \leq t$).

Termination: From Termination of A and $v_i \in \mathcal{H}$, $\exists v \in \mathcal{V}_i$ and v decides, so does v_i in B.

Validity: From Validity in A.

Consistency: From Consistency in A.

□

Bit Complexity

Theorem 3.3 (Dolev, Reischuk 1985).

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$$\sigma' \overset{v}{\sim} \sigma_0 \Rightarrow decision_v(\sigma') = 0,\ \ and$$
$$\sigma' \overset{u}{\sim} \sigma_1 \Rightarrow decision_u(\sigma') = 1,\quad \forall u \in \{\mathcal{H} \setminus \{v\}\}$$

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$$\sigma' \overset{u}{\sim} \sigma_1 \Rightarrow decision_u(\sigma') = 1,\ \ \forall u \in \{\mathcal{H} \setminus \{v\}\}$$

Hence $|A(v)| \geq t+1 \Rightarrow n(t+1)/2$ overall messages in both scenarios

$\Rightarrow$ At least $n(t+1)/4$ messages in σ_0 or σ_1. □

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