Άσκηση 8, φυλλάδιο "ΑΣΚ. Θ. GAUSS"

Ροή του $\vec{F} = f \nabla f$ διαμέσου της S (λόγω Θ. Gauss) είναι

$$\iiint_K \operatorname{div} \vec{F}.$$

$$\nabla f = (f_x, f_y, f_z) \Rightarrow \vec{F} = (f f_x, f f_y, f f_z)$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(f f_x) + \frac{\partial}{\partial y}(f f_y) + \frac{\partial}{\partial z}(f f_z)$$

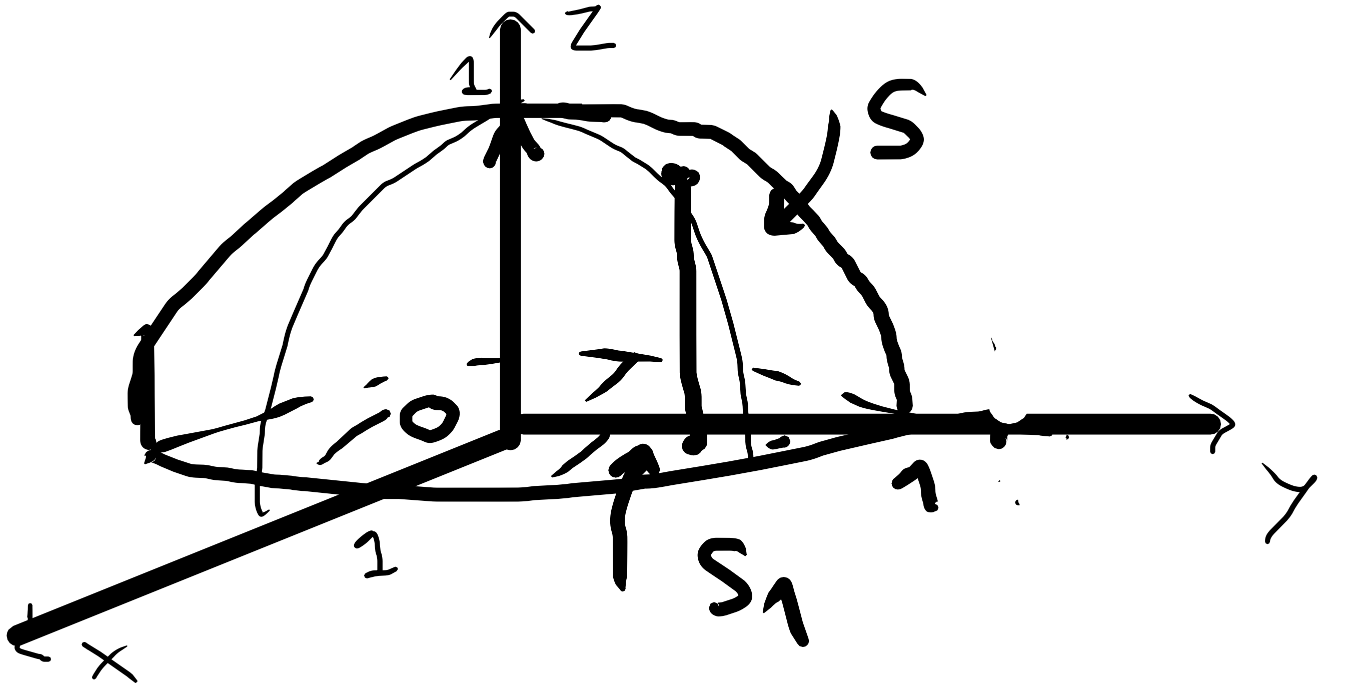
$$= f_x^2 + f_{xx} + f_y^2 + f_{yy} + f_z^2 + f_{zz} = |\nabla f|^2 \dots$$

<u>Άσκηση</u>: Δίνεται το διαν. πεδίο

$$\vec{F} = (x^2 e^z + x,\ y - x e^z,\ x^2 - z x e^z).$$

Να βρείτε τη ροή του $\vec{F}$ διαμέσου της

$$S = \{ (x,y,z) : x^2 + y^2 + z^2 = 1,\ z \geq 0 \}.$$




Θέτω $S_2 = S^* \cup S_1^*$,

$S_1^*$: $x^2 + y^2 \leq 1$, $z = 0$

ροή διαμέσου της $S =$

$=$ ροή διαμ. $(S_2)$ $-$ ροή διαμ. $(S_1)$

$S_2$ <u>κλειστή</u>! $\overset{\Theta.\text{Gauss}}{\Longrightarrow}$ ροή διατ. $(S_2) =$

$$= \iiint_K \operatorname{div}\vec{F} \quad , \quad K = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$$

$$= \iiint_K \left(\cancel{2xe^z} + 1 + 1 - \cancel{2xe^z}\right) dx\,dy\,dz =$$

$$= 2\iiint_K dx\,dy\,dz = 2\iint_{x^2+y^2\le 1} \left( \int_0^{\sqrt{1-x^2-y^2}} dz \right) dx\,dy$$

$$= 2 \iint_{x^2+y^2 \leq 1} \sqrt{1-x^2-y^2}\, dx\, dy \overset{\text{πολικές}}{=} 2 \int_0^{2\pi} \int_0^1 \sqrt{1-r^2}\, r\, dr\, d\varphi$$

$$= 4\pi \int_0^1 \sqrt{1-r^2}\, r\, dr \overset{t=1-r^2}{=} -2\pi \int_1^0 \sqrt{t}\, dt$$

$$= 2\pi \int_0^1 \sqrt{t}\, dt = 2\pi \, \frac{2}{3}\, t^{3/2} \Big|_0^1 = 4\pi/3$$

$S_1 : \Delta \to \mathbb{R}^3$, $S_1(u,v) = (u,v,0)$, $\Delta = \{(u,v) : u^2+v^2 \leq 1\}$

$$\frac{\partial S_1}{\partial u} \times \frac{\partial S_1}{\partial v} = (0,0,1) \qquad \vec{n}(u,v) = (0,0,-1)$$

Pon fl af. $(S_1) = \iint\limits_{\Delta} \vec{F}(S_1(u,v)) \cdot \vec{n}(u,v)\,du\,dv$

$$= \iint\limits_{\Delta} (u^2+u,\; v-u,\; u^2-2u) \cdot (0,0,-1)\,du\,dv$$

$$= \iint\limits_{\Delta} (2u-u^2)\,du\,dv = \underbrace{2\int_0^{2\pi}\int_0^1 r^2\cos\varphi\,dr\,d\varphi}_{0} -$$

$$- \int_0^{2\pi}\int_0^1 r^3\cos^2\varphi\,dr\,d\varphi = -\frac{1}{4}\int_0^{2\pi}\cos^2\varphi\,d\varphi =$$

$$= \dots = \underline{\underline{-\pi/4}}$$

Τελικά, η ροή διαμ. της $S$ =

$$= \frac{4\pi}{3} + \frac{\pi}{4} = \underline{\underline{\frac{19\pi}{12}}}.$$